DELIGNE PAIRINGS AND THE KNUDSEN-MUMFORD EXPANSION

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Abstract

Let $X \to B$ be a proper flat morphism between smooth quasi-projective varieties of relative dimension n, and $L \to X$ a line bundle which is ample on the fibers. We establish formulas for the first two terms in the Knudsen-Mumford expansion for $\det(\pi_* L^k)$ in terms of Deligne pairings of L and the relative canonical bundle K. This generalizes the theorem of Deligne [**1**], which holds for families of relative dimension one. As a corollary, we show that when X is smooth, the line bundle η associated to $X \to B$, which was introduced in Phong-Sturm [**12**], coincides with the CM bundle defined by Paul-Tian [**10**, **11**]. In a second and third corollaries, we establish asymptotics for the K-energy along Bergman rays generalizing the formulas obtained in [**11**].

1. Introduction

Let $\pi : X \to B$ be a flat proper morphism of integral schemes with constant relative dimension n, and let $L \to X$ be a relatively ample line bundle. The theorem of Knudsen-Mumford [**6**] says that there exist functorially defined line bundles $\lambda_j = \lambda_j(X, L, B) \to B$ with the property:

$$(1.1) \qquad \det \pi_*(L^k) \ \approx \ \lambda_{n+1}^{\binom{k}{n+1}} \otimes \lambda_n^{\binom{k}{n}} \otimes \cdots \otimes \lambda_0 \quad \text{for } k \gg 0.$$

In the case $n = 1$, Deligne [**1**] showed that $\lambda_2(L, X, B) = \langle L, L \rangle_{X/B}$, the Deligne pairing of L with itself. If in addition the varieties X and B are smooth, Deligne proved that $\lambda_1(L, X, B)^2 = \langle LK^{-1}, L \rangle_{X/B}$, where $K = K_{X/B} = K_X \otimes K_B^{-1}$ is the relative canonical line bundle. Our first result provides a generalization of these formulas to the case where $n \geq 0$:

Theorem 1. *Let $\pi : X \to B$ be a proper flat morphism of integral schemes of relative dimension $n \geq 0$ and let $L \to X$ be a line bundle which is very ample on the fibers.*