J. DIFFERENTIAL GEOMETRY
78 (2008) 269-293

HARMONIC MAPS FROM A SIMPLICIAL COMPLEX AND GEOMETRIC RIGIDITY

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Abstract

We study harmonic maps from an admissible flat simplicial complex to a non-positively curved Riemannian manifold. Our main regularity theorem is that these maps are $C^{1,\beta}$ at the interfaces of the top-dimensional simplices in addition to satisfying a balancing condition. If we assume that the domain is a 2-complex, then these maps are C^∞. As an application, we show that the regularity, the balancing condition and a Bochner formula lead to rigidity and vanishing theorems for harmonic maps. Furthermore, we give an explicit relationship between our techniques and those obtained via combinatorial methods.

1. Introduction

Harmonic maps are critical points of the energy functional. The energy of a map $\varphi : X \to N$ between two spaces X and N is defined to be the integral over the domain space of the energy density function which measures the total stretch of the map at each point of X. In the case when X and N are smooth Riemannian manifolds, the energy density function is the squared norm of the differential of the map.

One of the highlights of the harmonic map theory has been its successful applications to study representation of discrete groups. Suppose Γ is a fundamental group of a manifold X acting on a space N by $\rho : \Gamma \to \mathrm{Isom}(N)$. The idea is to associate the action with an equivariant harmonic map $\tilde{f} : \tilde{X} \to N$ where $\tilde{X}$ is the universal cover of X. Once the existence is established, one can use the curvature assumptions on the domain and the target spaces to make strong statements about $\tilde{f}$ and hence about the representation ρ.

To illustrate this, let X be a compact Riemannian manifold of non-negative Ricci curvature and with fundamental group Γ, and N be a complete Riemannian manifold of non-positive sectional curvature. Consider a representation $\rho : \Gamma \to \mathrm{Isom}(N)$ and let $\tilde{f} : \tilde{X} \to N$ be a

The first author was partially supported by research grant NSF DMS-0604930. The second author was partially supported by research grant NSF DMS-0450083.
Received 09/10/2005.