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THE YAMABE PROBLEM FOR HIGHER ORDER CURVATURES

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Abstract

Let $\mathcal{M}$ be a compact Riemannian manifold of dimension $n > 2$. The k-curvature, for $k = 1, 2, \ldots, n$, is defined as the k-th elementary symmetric polynomial of the eigenvalues of the Schouten tenser. The k-Yamabe problem is to prove the existence of a conformal metric whose k-curvature is a constant. When $k = 1$, it reduces to the well-known Yamabe problem. Under the assumption that the metric is admissible, the existence of solutions is known for the case $k = 2$, $n = 4$, for locally conformally flat manifolds and for the cases $k > n/2$. In this paper we prove the solvability of the k-Yamabe problem in the remaining cases $k \leq n/2$, under the hypothesis that the problem is variational. This includes all of the cases $k = 2$ as well as the locally conformally flat case.

1. Introduction

In recent years the Yamabe problem for the k-curvature of the Schouten tensor, or simply the k-Yamabe problem, has been extensively studied. Let $(\mathcal{M}, g_0)$ be a compact Riemannian manifold of dimension $n > 2$ and denote by 'Ric' and R respectively the Ricci tensor and the scalar curvature. The k-Yamabe problem is to prove the existence of a conformal metric $g = g_v = v^{\frac{4}{n-2}} g_0$ that solves the equation

$$\sigma_k(\lambda(A_g)) = 1 \quad \text{on} \quad \mathcal{M}, \tag{1.1}$$

where $1 \leq k \leq n$ is an integer, and $\lambda = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of A_g with respect to the metric g. As usual, we denote by

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)} g\right) \tag{1.2}$$

the Schouten tensor, and by

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \tag{1.3}$$

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