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A STRUCTURE THEOREM FOR THE GROMOV-WITTEN INVARIANTS OF KÄHLER SURFACES

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Abstract

We prove a structure theorem for the Gromov-Witten invariants of compact Kähler surfaces with geometric genus $p_g > 0$. Under the technical assumption that there is a canonical divisor that is a disjoint union of smooth components, the theorem shows that the GW invariants are universal functions determined by the genus of this canonical divisor components and the holomorphic Euler characteristic of the surface. We compute special cases of these universal functions.

Much of the work on the Gromov-Witten invariants of Kähler surfaces has focused on rational and ruled surfaces, which have geometric genus $p_g = 0$. This paper focuses on surfaces with $p_g > 0$, a class that includes most elliptic surfaces and most surfaces of general type. In this context we prove a general "structure theorem" that shows (with one technical assumption) how the GW invariants are completely determined by the local geometry of a generic canonical divisor.

The structure theorem is a consequence of a simple fact: the "Image Localization Lemma" of Section 3. Given a Kähler surface X and a canonical divisor $D \in |K_X|$, this lemma shows that the complex structure J on X can be perturbed to a *non-integrable* almost complex structure J_D with the property that the image of all J_D-holomorphic maps lies in the support of D. This immediately gives some striking vanishing theorems for the GW invariants of Kähler surfaces (see Section 3). More importantly, it implies that the Gromov-Witten invariant of X for genus g and n marked points is a sum

$$GW_{g,n}(X, A) = \sum GW^{\mathrm{loc}}_{g,n}(D_k, A_k)$$

over the connected components D_k of D of "local invariants" that count the contribution of maps whose image lies in or (after perturbing to a generic moduli space) near D_k. These local invariants have not been previously defined. The proof of their existence relies on using non-integrable structures and geometric analysis techniques.

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