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ARAKELOV INEQUALITIES AND THE UNIFORMIZATION OF CERTAIN RIGID SHIMURA VARIETIES

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Abstract

Let Y be a non-singular projective manifold with an ample canonical sheaf, and let $\mathbb{V}$ be a $\mathbb{Q}$-variation of Hodge structures of weight one on Y with Higgs bundle $E^{1,0} \oplus E^{0,1}$, coming from a family of Abelian varieties. If Y is a curve the Arakelov inequality says that the slopes satisfy $\mu(E^{1,0}) - \mu(E^{0,1}) \leq \mu(\Omega^1_Y)$.

We prove a similar inequality in the higher dimensional case. If the latter is an equality, and if the discriminant of $E^{1,0}$ or the one of $E^{0,1}$ is zero, one hopes that Y is a Shimura variety, and $\mathbb{V}$ a uniformizing variation of Hodge structures. This is verified, in case the universal covering of Y does not contain factors of rank > 1. Part of the results extend to variations of Hodge structures over quasi-projective manifolds U.

Let Y be a complex n-dimensional projective manifold, $S \subset Y$ a reduced normal crossing divisor, $U = Y \setminus S$, and let $f : V \to U$ be a smooth family of g-dimensional Abelian varieties. Assume that the local system $R^1 f_* \mathbb{C}_V$ has uni-potent monodromy along the components of S. Let $\mathbb{V}$ be a C-sub-variation of Hodge structures in $R^1 f_* \mathbb{C}_V$.

The Deligne extension of $\mathbb{V} \otimes \mathcal{O}_U$ to Y carries a Hodge filtration. Taking the graded sheaf one obtains the (logarithmic) Higgs bundle

$$(E, \theta) = (E^{1,0} \oplus E^{0,1}, \theta),$$

where $\theta : E \to E \otimes \Omega^1_Y(\log S)$ is zero on $E^{0,1}$ and factors through

$$\theta : E^{1,0} \longrightarrow E^{0,1} \otimes \Omega^1_Y(\log S)$$

on $E^{1,0}$. Define for a torsion free coherent sheaf $\mathcal{F}$ on Y

$$\Upsilon(\mathcal{F}) = \frac{\mathrm{c}_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} \in H^2(Y, \mathbb{Q}) \quad \text{and}$$

$$\Delta(\mathcal{F}) = 2 \cdot \mathrm{rk}(\mathcal{F}) \cdot \mathrm{c}_2(\mathcal{F}) - (\mathrm{rk}(\mathcal{F}) - 1) \cdot \mathrm{c}_1(\mathcal{F})^2 \in H^4(Y, \mathbb{Q}).$$

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