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A DIFFEOMORPHISM CLASSIFICATION OF MANIFOLDS WHICH ARE LIKE PROJECTIVE PLANES

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Abstract

We give a complete diffeomorphism classification of 1-connected closed manifolds M with integral homology $H_*(M) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, provided that $\dim(M) \neq 4$.

The integral homology of an oriented closed manifold[1] M contains at least two copies of $\mathbb{Z}$ (in degree 0 resp. $\dim M$). If M is simply connected and its homology has minimal size (i.e., $H_*(M) \cong \mathbb{Z} \oplus \mathbb{Z}$), then M is a homotopy sphere (i.e., M is homotopy equivalent to a sphere). It is well-known from the proof of the (generalized) Poincaré conjecture that any homotopy sphere is homeomorphic to the standard sphere S^n of dimension n. By contrast, the cardinality of the set Θ_n of diffeomorphism classes of homotopy spheres of dimension n can be very large (but finite except possibly for $n = 4$) [**7**]. In fact, the connected sum of homotopy spheres gives Θ_n the structure of an abelian group which is closely related to the stable homotopy group $\pi_{n+k}(S^k)$, $k \gg n$ (currently known approximately in the range $n \leq 100$).

Somewhat surprisingly, it is easier to obtain an explicit diffeomorphism classification of 1-connected closed manifolds whose integral homology consists of *three* copies of $\mathbb{Z}$. Examples of such manifolds are the 1-connected projective planes (i.e., the projective planes over the complex numbers, the quaternions or the octonions). Eells and Kuiper pioneered the study of these 'projective plane like' manifolds [**4**] and obtained many important and fundamental results. For example, they show that the integral cohomology ring of such a manifold M is isomorphic to the cohomology ring of a projective plane, i.e., $H^*(M) \cong \mathbb{Z}[x]/(x^3)$. This in turn implies that the dimension of M must be $2m$ with $m = 2, 4$ or 8 (cf. [**4**, §5]).

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[1] All manifolds are assumed to be smooth.