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LOGARITHMIC GROWTH OF SYSTOLE OF ARITHMETIC RIEMANN SURFACES ALONG CONGRUENCE SUBGROUPS

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Abstract

We apply a study of orders in quaternion algebras, to the differential geometry of Riemann surfaces. The least length of a closed geodesic on a hyperbolic surface is called its systole, and denoted $\mathrm{sys}\pi_1$. P. Buser and P. Sarnak constructed Riemann surfaces X whose systole behaves logarithmically in the genus $g(X)$. The Fuchsian groups in their examples are principal congruence subgroups of a fixed arithmetic group with rational trace field. We generalize their construction to principal congruence subgroups of arbitrary arithmetic surfaces. The key tool is a new trace estimate valid for an arbitrary ideal in a quaternion algebra. We obtain a particularly sharp bound for a principal congruence tower of Hurwitz surfaces (PCH), namely the 4/3-bound $\mathrm{sys}\pi_1(X_{\mathrm{PCH}}) \geq \frac{4}{3}\log(g(X_{\mathrm{PCH}}))$. Similar results are obtained for the systole of hyperbolic 3-manifolds, relative to their simplicial volume.

1. Orders in quaternion algebras and Riemann surfaces

Arithmetic lattices, besides their own intrinsic interest, have traditionally provided a rich source of examples in geometry. One striking application is the construction of isospectral, non-isometric hyperbolic surfaces by M.-F. Vigneras [**Vig80**]. A survey of arithmeticity as applied in geometry and dynamics may be found in [**Pa95**]. See [**Lub94**] for an application of congruence subgroups and the literature on girth in graph theory initiated by W. Tutte [**Tu47**]. See also [**ChW06**] for a recent geometric application of congruence subgroups.

While the simplest definition of arithmeticity, in analogy with $\mathrm{SL}_2(\mathbb{Z})$, can be presented in terms of n-dimensional representations by matrices defined over the integers, for many purposes it is convenient to work with

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