J. DIFFERENTIAL GEOMETRY
76 (2007) 317-349

THE CALABI-YAU EQUATION ON ALMOST-KÄHLER FOUR-MANIFOLDS

Ben Weinkove

Abstract

Let (M, ω) be a compact symplectic 4-manifold with a compatible almost complex structure J. The problem of finding a J-compatible symplectic form with prescribed volume form is an almost-Kähler analogue of Yau's theorem and is connected to a programme in symplectic topology proposed by Donaldson. We call the corresponding equation for the symplectic form the *Calabi-Yau equation*. Solutions are unique in their cohomology class. It is shown in this paper that a solution to this equation exists if the Nijenhuis tensor is small in a certain sense. Without this assumption, it is shown that the problem of existence can be reduced to obtaining a C^0 bound on a scalar potential function.

1. Introduction

In 1954 Calabi [**Ca**] conjectured that any representative of the first Chern class of a compact Kähler manifold (M, ω) can be written as the Ricci curvature of a Kähler metric ω' cohomologous to ω. He showed that any such metrics are unique. Yau [**Ya**] famously solved Calabi's conjecture around twenty years later. This result, and the immediate corollary that any Kähler manifold with $c_1(M) = 0$ admits a Ricci-flat metric, have had many applications in both mathematics and theoretical physics.

Yau's theorem is equivalent to finding a Kähler metric in a given Kähler class with prescribed volume form. By the $\partial\overline{\partial}$-Lemma this amounts to solving the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}\phi)^n = e^F \omega^n, \tag{1.1}$$

for smooth real ϕ with $\omega + \sqrt{-1}\partial\overline{\partial}\phi > 0$, where $n = \dim_{\mathbb{C}} M$ and F is any smooth function with $\int_M e^F \omega^n = \int_M \omega^n$. Yau solved this equation by considering the family of equations obtained by replacing F by $tF + c_t$ for some constant c_t, for $t \in [0, 1]$ and using the continuity method. This

The author is supported in part by National Science Foundation grant DMS-05-04285. This work was carried out while the author was visiting Imperial College, London on a Royal Society Research Assistantship.

Received 05/09/2006.