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A SURGERY FOR GENERALIZED COMPLEX STRUCTURES ON 4-MANIFOLDS

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Abstract

We introduce a surgery for generalized complex manifolds whose input is a symplectic 4-manifold containing a symplectic 2-torus with trivial normal bundle and whose output is a 4-manifold endowed with a generalized complex structure exhibiting type change along a 2-torus. Performing this surgery on a $K3$ surface, we obtain a generalized complex structure on $3\mathbb{C}P^2\#19\overline{\mathbb{C}P^2}$, which has vanishing Seiberg–Witten invariants and hence does not admit complex or symplectic structures.

Introduction

Generalized complex structures, introduced by Hitchin [**3**] and developed by the second author in [**2**], are a simultaneous generalization of complex and symplectic structures. In this paper we answer, in the affirmative, the question of whether there exist manifolds which are neither complex nor symplectic yet do admit a generalized complex structure.

Since generalized complex manifolds must be almost complex, this question becomes nontrivial first in dimension 4, where we are fortunate to have obstructions to the existence of complex and symplectic structures coming from Seiberg–Witten theory. For example, a simply-connected complex or symplectic 4-manifold with $b_+ \geq 3$ must have a nonzero Seiberg–Witten invariant [**7**].

Each tangent space of a generalized complex manifold has a distinguished subspace equipped with a symplectic form and a transverse complex structure; the transverse complex dimension is called the *type*, a local invariant of the geometry which may vary along the manifold. We show that in 4 dimensions, a connected and nondegenerate type change locus must be a smooth 2-torus, which also inherits a complex structure, i.e. it must be a nonsingular elliptic curve.

We then introduce a surgery for generalized complex manifolds which is a particular case of the C^∞ *logarithmic transformation* introduced by Gompf and Mrowka [**1**]. This surgery modifies a neighbourhood

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