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SELF-DUAL METRICS AND TWENTY-EIGHT BITANGENTS

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Abstract

We determine a global structure of the moduli space of self-dual metrics on $3\mathbf{CP}^2$ satisfying the following three properties: (i) the scalar curvature is of positive type, (ii) they admit a non-trivial Killing field, (iii) they are not conformal to the LeBrun's self-dual metrics based on the 'hyperbolic ansatz'. We prove that the moduli space of these metrics is isomorphic to an orbifold $\mathbf{R}^3/G$, where G is an involution of $\mathbf{R}^3$ having two-dimensional fixed locus. In particular, the moduli space is non-empty and connected. We also remark that Joyce's self-dual metrics with torus symmetry appear as a limit of our self-dual metrics.

Our proof of the result is based on the twistor theory. We first determine a defining equation of a projective model of the twistor space of the metric, and then prove that the projective model is always birational to a twistor space, by determining the family of twistor lines. In determining them, a key role is played by a classical result in algebraic geometry that a smooth plane quartic always possesses twenty-eight bitangents.

1. Introduction

A Riemannian metric on an oriented four-manifold is called self-dual if the anti-self-dual part of the Weyl conformal curvature of the metric identically vanishes. Basic examples are provided by the round metric on the four-sphere and the Fubini-Study metric on the complex projective plane. In general, one can expect that if two four-manifolds admit self-dual metrics respectively, then their connected sum will also admit a self-dual metric. In fact, Y.S. Poon [**15**] constructed explicit examples of self-dual metrics on $2\mathbf{CP}^2$, the connected sum of two complex projective planes. He further showed that on $2\mathbf{CP}^2$ there are no self-dual metrics other than his metrics, under assumption of the positivity of the scalar curvature. Later, C. LeBrun [**11**] and D. Joyce [**8**] respectively constructed a family of self-dual metrics of positive scalar curvature

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