J. DIFFERENTIAL GEOMETRY
75 (2007) 57-108

BÄCKLUND TRANSFORMATIONS, WARD SOLITONS, AND UNITONS

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Abstract

The Ward equation, also called the modified $2+1$ chiral model, is obtained by a dimension reduction and a gauge fixing from the self-dual Yang-Mills field equation on $\mathbb{R}^{2,2}$. It has a Lax pair and is an integrable system. Ward constructed solitons whose extended solutions have distinct simple poles. He also used a limiting method to construct 2-solitons whose extended solutions have a double pole. Ioannidou and Zakrzewski, and Anand constructed more soliton solutions whose extended solutions have a double or triple pole. Some of the main results of this paper are: (i) We construct algebraic Bäcklund transformations (BTs) that generate new solutions of the Ward equation from a given one by an algebraic method. (ii) We use an order k limiting method and algebraic BTs to construct explicit Ward solitons, whose extended solutions have arbitrary poles and multiplicities. (iii) We prove that our construction gives all solitons of the Ward equation explicitly and the entries of Ward solitons must be rational functions in x, y and t. (iv) Since stationary Ward solitons are unitons, our method also gives an explicit construction of all k-unitons from finite sequences of rational maps from $\mathbb{C}$ to $\mathbb{C}^n$.

1. Introduction

The $2+1$ *chiral model* is the Euler-Lagrange equation of the functional

$$\mathcal{E}(J) = \int_{\mathbb{R}^3} \|J^{-1}J_x\|^2 + \|J^{-1}J_y\|^2 - \|J^{-1}J_t\|^2 \, dx\, dy\, dt,$$

where $\|\xi\|^2 = -\mathrm{tr}(\xi^2)$, x, y, t are the standard space-time variables, and J is a map from the Lorentz space $\mathbb{R}^{2,1}$ to the Lie group $U(n)$. In other words, J is a solution of

$$(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y = 0. \tag{1.1}$$

The research of the first author was supported in part by the AMS Fan Fund and a CNSF Grant under no. 10321001. The research of the second author was supported in part by NSF Grant DMS-0306446.
Received 12/02/2004.