J. DIFFERENTIAL GEOMETRY
74 (2006) 425-505

NONABELIAN JACOBIAN OF SMOOTH PROJECTIVE SURFACES

IGOR REIDER

Abstract

The paper proposes a nonabelian version of the Jacobian for a smooth complex projective surface X. Our version possesses all the classical features: it is the parameter space for a "canonical" family of torsion-free sheaves over X having fixed Chern invariants and rank 2 , it carries a distinguished divisor (a "theta-divisor"), a "package" of nonabelain "theta-functions". But it also has a new feature: our Jacobian carries a distinguished family of Higgs bundles. The parameter space H, called (nonabelian) Albanese, of this family is a projective toric (singular) Fano variety whose hyperplane sections are (singular) Calabi-Yau varieties. In particular, it comes with a distinguished degenerate hyperplane section H_0 equipped with degenerate symplectic structure, i.e., H_0 is the union of projectivized Lagrangian subspaces of a certain symplectic vector space naturally associated with H_0.

Our Jacobian and its Albanese H are related by two correspondences:

(i) a geometric correspondence which sends points of the nonabelian Jacobian to a cycle of Calabi-Yau varieties,

(ii) a cohomological correspondence, which is a Fourier-Mukai functor from the Higgs category on the Jacobian (algebraic/holomorphic side) to the so called F-category on H (algebraic/symplectic side).

Furthermore, there is a "quantum" correspondence which associates an operator-valued series with points of our Jacobian. The operator coefficients of this series are most naturally considered as elements of the universal enveloping algebra of a certain Lie algebra canonically associated to every point of the Jacobian. This gives a sheaf of Lie algebras on our Jacobian which could be viewed as a natural analogue of the Lie algebraic structure of the classical Jacobian.

The basic Lie algebraic properties of this sheaf are established and a dictionary between its representation theory and geometry of the underlying points is given.

Received 12/16/2004.