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ENLARGEABILITY AND INDEX THEORY

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Abstract

Let M be a closed enlargeable spin manifold. We show nontriviality of the universal index obstruction in the K-theory of the maximal C^*-algebra of the fundamental group of M. Our proof is independent of the injectivity of the Baum-Connes assembly map for $\pi_1(M)$ and relies on the construction of a certain infinite dimensional flat vector bundle out of a sequence of finite dimensional vector bundles on M whose curvatures tend to zero.

Besides the well known fact that M does not carry a metric with positive scalar curvature, our results imply that the classifying map $M \to B\pi_1(M)$ sends the fundamental class of M to a nontrivial homology class in $H_*(B\pi_1(M); \mathbb{Q})$. This answers a question of Burghelea (1983).

1. Introduction

1.1. Enlargeability and the universal index obstruction. For a closed spin manifold M^n, Rosenberg in [**16**] constructs an index

$$\alpha^{\mathbb{R}}_{\max}(M) \in KO_n(C^*_{\max,\mathbb{R}}\pi_1(M))$$

in the K-theory of the (maximal) real C^*-algebra of the fundamental group of M. By the Lichnerowicz-Schrödinger-Weitzenböck formula this index is zero if M admits a metric of positive scalar curvature. The Gromov-Lawson-Rosenberg conjecture states that, conversely, the vanishing of $\alpha(M)$ implies that M admits such a metric, if $n \geq 5$. By a result of the second named author, this conjecture is known to be false in general [**17**] or [**4**]. But a stable version of this conjecture is true, if the Baum-Connes assembly map

$$\mu\colon KO_*^{\pi_1(M)}(\underline{E}\pi_1(M)) \to KO_*(C^*_{\max,\mathbb{R}}\pi_1(M))$$

is injective [**20**]. The proof of this (and related results) is based on the existence of a natural map $D\colon KO_*(M) \to KO_*^{\pi_1(M)}(\underline{E}\pi_1(M))$ into the

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