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Higher Bruhat and Tamari Orders and their Realizations

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Abstract

We briefly describe recent results about higher Bruhat and Tamari orders, the associated simplex equations and generalizations (polygon equations) of the pentagon equation, and the appearance of these orders in soliton interactions.

Keywords: Associahedron; Higher Bruhat order; KP equation; Pentagon equation; Permutahedron; Simplex equation; Soliton; Tamari order; Yang-Baxter equation

Introduction

In this letter, we sketch some of our recent results [1] at the interface between integrable systems and combinatorics. This includes a revision of the relation between *higher Bruhat orders* and *simplex equations* [2,3], a decomposition of higher Bruhat orders, the resulting Tamari orders (expected to be equivalent to higher Stasheff -Tamari orders), and a new family of equations associated with the latter. We finally recall the occurrence of higher Bruhat and Tamari orders in a "tropical limit" of solitons of the famous Kadomtsev-Petviashvili (KP) equation [4,5].

Higher Bruhat Orders and Simplex Equations

Let $[N] = \{1, 2, \ldots, N\}$ and $\binom{[N]}{n}$ be the set of n-element subsets of $[N]$. A linear order (permutation) of $\binom{[N]}{n}$ is called *admissible* if for any $K \in \binom{[N]}{n+1}$, the packet $P(K) :=$ {n-element subsets of K} is contained in it in *lexicographical* $(\vec{P}(K))$ or in *reverse lexicographical* order $(\overleftarrow{P}(K))$. Let $A(N, n)$ be the set of admissible linear orders of $\binom{[N]}{n}$. An equivalence relation is defined on $A(N, n)$ by setting $\rho \sim \rho'$ if ρ and ρ' only differ by exchange of two neighboring elements, not both contained in some packet. The *higher Bruhat order* $B(N, n)$ is $A(N, n)/\sim$ supplied with the partial order determined by inversions $\vec{P}(K) \mapsto \overleftarrow{P}(K)$ of lexicographically ordered packets of neighboring elements. There is a natural correspondence between the elements of $A(N, n + 1)$ and the maximal chains of $B(N, n)$. The Bruhat order $B(3, 1)$ consists of the two maximal chains

$$(1,2,3) \xrightarrow{12} (2,1,3) \xrightarrow{13} (2,3,1) \xrightarrow{23} (3,2,1)$$
$$(1,2,3) \xrightarrow{23} (1,3,2) \xrightarrow{13} (3,1,2) \xrightarrow{12} (3,2,1)$$

where ij stands for $\{i, j\}$ and here indicates the inversion of the packet of it in the respective linear order. A set-theoretical realization

$$i \mapsto \mathcal{U}_i, \quad (i,j,k) \mapsto \mathcal{U}_i \times \mathcal{U}_j \times \mathcal{U}_k, \quad ij \mapsto \mathcal{R}_{ij} : \mathcal{U}_i \times \mathcal{U}_j \to \mathcal{U}_j \times \mathcal{U}_i$$

(or a realization using vector spaces and tensor products) leads to the *Yang-Baxter (YB) equation*

$$\mathcal{R}_{23,\mathbf{12}} \mathcal{R}_{13,\mathbf{23}} \mathcal{R}_{12,\mathbf{12}} = \mathcal{R}_{12,\mathbf{23}} \mathcal{R}_{13,\mathbf{12}} \mathcal{R}_{23,\mathbf{23}}$$

The boldface *position indices* are read off from the above chains. They specify on which pair of sets in the threefold direct product the map $\mathcal{R}_{ij}$ acts. In terms of $\hat{\mathcal{R}}_{ij} := \mathcal{R}_{ij}\mathcal{P}$, where $\mathcal{P}$ is the transposition, the YB equation has the more familiar form $\hat{\mathcal{R}}_{23,\mathbf{12}} \hat{\mathcal{R}}_{13,\mathbf{13}} \hat{\mathcal{R}}_{12,\mathbf{23}} = \hat{\mathcal{R}}_{12,\mathbf{23}} \hat{\mathcal{R}}_{13,\mathbf{13}} \hat{\mathcal{R}}_{23,\mathbf{12}}$.

For any N>1, the higher Bruhat order $B(N+1, N-1)$ consists of two maximal chains, so that it determines an equation in the same way as in the above example, where N=2. The associated equation is the *N-simplex equation*. The 3-simplex equation is thus a realization of $B(4, 2)$. The two maximal chains of the latter poset are resolved below into elements of $A(4, 2)$, and a linear order is now more conveniently displayed as a column. The coloring is referred to later on.

$$\begin{array}{c} 12\\13\\14\\23\\24\\34 \end{array} \overset{\sim}{\longrightarrow} \begin{array}{c} 12\\13\\23\\14\\24\\34 \end{array} \xrightarrow{123} \begin{array}{c} 23\\13\\12\\14\\24\\34 \end{array} \xrightarrow{124} \begin{array}{c} 23\\13\\24\\14\\12\\34 \end{array} \overset{\sim}{\longrightarrow} \begin{array}{c} 23\\24\\13\\14\\34\\12 \end{array} \xrightarrow{134} \begin{array}{c} 23\\24\\34\\14\\13\\12 \end{array} \xrightarrow{234} \begin{array}{c} 34\\24\\23\\14\\13\\12 \end{array}$$

$$\begin{array}{c} 12\\13\\14\\23\\24\\34 \end{array} \xrightarrow{234} \begin{array}{c} 12\\13\\14\\34\\24\\23 \end{array} \xrightarrow{134} \begin{array}{c} 12\\34\\14\\13\\24\\23 \end{array} \overset{\sim}{\longrightarrow} \begin{array}{c} 34\\12\\14\\24\\13\\23 \end{array} \xrightarrow{124} \begin{array}{c} 34\\24\\14\\12\\13\\23 \end{array} \xrightarrow{123} \begin{array}{c} 34\\24\\14\\23\\13\\12 \end{array} \overset{\sim}{\longrightarrow} \begin{array}{c} 34\\24\\23\\14\\13\\12 \end{array}$$

From the above chains we read off the 3-simplex equation

$$\mathcal{R}_{234,\mathbf{123}} \mathcal{R}_{134,\mathbf{345}} \mathcal{P}_{56} \mathcal{P}_{23} \mathcal{R}_{124,\mathbf{345}} \mathcal{R}_{123,\mathbf{123}} \mathcal{P}_{\mathbf{34}} = \mathcal{P}_{\mathbf{34}} \mathcal{R}_{123,\mathbf{456}} \mathcal{R}_{124,\mathbf{234}} \mathcal{P}_{\mathbf{45}} \mathcal{P}_{\mathbf{12}} \mathcal{R}_{134,\mathbf{234}} \mathcal{R}_{234,\mathbf{456}}$$

for maps $\mathcal{R}_{ijk} : \mathcal{U}_{ij} \times \mathcal{U}_{ik} \times \mathcal{U}_{jk} \to \mathcal{U}_{jk} \times \mathcal{U}_{ik} \times \mathcal{U}_{ij}$, $i < j < k$. An equivalence ($\sim$) is realized by a transposition map $\mathcal{P}_{ab}$ (acting at positions $\boldsymbol{a}$ and $\boldsymbol{b}$). In terms of $\hat{\mathcal{R}}_{ijk} := \mathcal{R}_{ijk}\mathcal{P}_{13} : \mathcal{U}_{jk} \times \mathcal{U}_{ik} \times \mathcal{U}_{ij} \to \mathcal{U}_{jk} \times \mathcal{U}_{ik} \times \mathcal{U}_{ij}$ the 3-simplex equation takes the form

$$\hat{\mathcal{R}}_{\hat{1},\mathbf{123}} \hat{\mathcal{R}}_{\hat{2},\mathbf{145}} \hat{\mathcal{R}}_{\hat{3},\mathbf{246}} \hat{\mathcal{R}}_{\hat{4},\mathbf{356}} = \hat{\mathcal{R}}_{\hat{4},\mathbf{356}} \hat{\mathcal{R}}_{\hat{3},\mathbf{246}} \hat{\mathcal{R}}_{\hat{2},\mathbf{145}} \hat{\mathcal{R}}_{\hat{1},\mathbf{123}}$$

using complementary notation: $\hat{1} = \{2,3,4\} = 234$ etc. This is also known as *tetrahedron equation*, or *Zamolodchikov equation*. It appeared originally as a factorization condition for the S-matrix in a (2+1)-dimensional theory of "straight strings", and in a related three-dimensional exactly solvable lattice model (see, e.g., [6]).

The six linear orders constituting $A(3, 1)$, viewed as maximal chains, build $B(3, 0)$. It forms a cube, on which the structure of the YB equation is often visualized. There is a counterpart for any simplex equation. For example, the linear orders constituting $A(4, 2)$ can be viewed as maximal chains of $B(4, 1)$, which forms a polyhedron called

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