TOKYO J. MATH.
VOL. 2, NO. 1, 1979

A Note on the Pell Equation

Hideo WADA

Sophia University

For solving the Pell equation $|x^2-my^2|=1$, we usually use the continued fraction expansion of $\sqrt{m}$. We will give here a new geometrical interpretation of the continued fraction expansion, and apply it to solve the Pell equation. Theorem 1 makes the continued fraction expansion of $\sqrt{m}$ more meaningful. Theorem 2 gives the existence of the solution. The proof is simpler and shorter than the usual one.

§1. Let m be a positive integer which is not a square. Then the Pell equation

$$(1) \qquad |x^2-my^2|=1\ , \quad (x,\ y\in \boldsymbol{Z})$$

can be written as

$$(2) \qquad |(x+\sqrt{m}\,y)(x-\sqrt{m}\,y)|=1\ .$$

Put $\alpha=x+\sqrt{m}\,y$, $\alpha'=x-\sqrt{m}\,y=$the conjugate of α. Then (2) can be written as $|\alpha\alpha'|=1$. Put $\alpha_0=1$, $\alpha_{-1}=\sqrt{m}$, $L=\{x\alpha_0+y\alpha_{-1}\,|\,x,\ y\in \boldsymbol{Z}\}=\{x+\sqrt{m}\,y\,|\,x,\ y\in \boldsymbol{Z}\}$, and in the X-Y plane, $\vec{\alpha}_0=(1,1)$, $\vec{\alpha}_{-1}=(\sqrt{m},\ -\sqrt{m})$, $\vec{L}=\{x\vec{\alpha}_0+y\vec{\alpha}_{-1}\,|\,x,\ y\in \boldsymbol{Z}\}=\{(x+\sqrt{m}\,y,\ x-\sqrt{m}\,y)\,|\,x,\ y\in \boldsymbol{Z}\}=\{(\alpha,\ \alpha')\,|\,\alpha\in L\}$.

LEMMA 1. *Let $\vec{\beta}=(\beta,\beta')$, $\vec{\gamma}=(\gamma,\gamma')$ be generators of $\vec{L}$ such that $0<\beta$, $0<\gamma$, $\beta'\gamma'<0$, $|\gamma'|<|\beta'|$. Then the smallest number $\delta\in L$ such that $\gamma<\delta$, $|\delta'|<|\gamma'|$ is $\beta+[-\beta'/\gamma']\gamma$, ($[-\beta'/\gamma']$ means the integer part of $-\beta'/\gamma'$). In this case, $\vec{\gamma}$, $\vec{\delta}=(\delta,\delta')$ are generators of $\vec{L}$ such that $0<\gamma$, $0<\delta$, $\gamma'\delta'<0$, $|\delta'|<|\gamma'|$.*

PROOF. We may assume $0<\gamma'<-\beta'$ without any loss of generality. Put $\delta=x\gamma+y\beta$. If $x\leqq 0$ and $y\leqq 0$, then $\delta\leqq 0$. If $x>0$ and $y\leqq 0$, then $\delta'=x\gamma'+y\beta'\geqq\gamma'$. Therefore y must be greater than zero. When $y\geqq 1$, then from the condition $|\delta'|<|\gamma'|$, we have $-\gamma'<x\gamma'+y\beta'<\gamma'$ and $-\gamma'-y\beta'<x\gamma'<\gamma'-y\beta'$. Hence $-1-\beta'/\gamma'\leqq -1-y\beta'/\gamma'<x<1-y\beta'/\gamma'$. From

Received March 3, 1978