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Formula for the Casimir Operator in Iwasawa Coördinates

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Introduction

Let G be a connected non-compact semisimple Lie group with an Iwasawa decomposition $G=KAN$, where the Lie algebra of K is $\mathfrak{k}$ in the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G, the Lie algebra $\mathfrak{a}$ of A is a maximal abelian subspace of $\mathfrak{p}$, and the Lie algebra $\mathfrak{n}$ of N is a sum of root spaces corresponding to a choice of positive *restricted* roots. Let Ω be the Casimir operator of G. We give a formula for Ω in terms of first order differential operators $\delta_z, \delta_H, \delta_x$ defined for left-invariant vector fields $(z, H, x)\in\mathfrak{k}\times\mathfrak{a}\times\mathfrak{n}$. Theorem 1.10 is the main result. Such a formula was first proposed in [1]. The arguments given there, however, are incomplete. Corrections are made in the present paper. In particular our formula reduces to Takahashi's formula [3] when $G=SL(2, \boldsymbol{R})$.

§1. Statement of the result.

Let $\mathfrak{g}$ be a non-compact real semisimple Lie algebra with a Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, and Cartan involution θ. The Killing form B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. The formula

$$(1.1)\qquad \langle x, y\rangle=-B(x, \theta y)\qquad x, y\in\mathfrak{g}$$

defines a real positive definite inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Let $\mathfrak{a}\subset\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$. For $\alpha\in\mathfrak{a}^*$ (the real dual space of $\mathfrak{a}$) let $\mathfrak{g}_\alpha=\{x\in\mathfrak{g}\mid[H, x]=\alpha(H)x$ for every H in $\mathfrak{a}\}$. α is a *restricted* root (relative to $\mathfrak{a}$) if both α and the root space $\mathfrak{g}_\alpha$ are non-zero. Let $\Sigma\subset\mathfrak{a}^*$ denote the set of restricted roots, let $\Sigma^+\subset\Sigma$ denote a choice of a system of positive restricted roots, let $\mathfrak{n}$ denote the sum of the positive root spaces, and let (for $\alpha\in\Sigma$)

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