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Ring Derivations on Semi-Simple Commutative Banach Algebras

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Introduction.

Let A be a commutative Banach algebra. An (resp. linear) operator D on A is called a *ring* (resp. *linear*) *derivation* on A if equations $D(f+g)=D(f)+D(g)$ and $D(fg)=fD(g)+D(f)g$ are satisfied for every f and g in A. The image of linear derivation was studied by Singer and Wermer [5] under the hypothesis of continuity of the operator, and Thomas [6] has proved that every linear derivation on a commutative Banach algebra maps into the radical of the algebra. On the other hand there are ring derivations which do not map into the radical (cf. [1]). In this paper we characterize ring derivations on semi-simple commutative Banach algebras. A function algebra is semi-simple and so the results generalize our previous results in [3]. As a consequence of the results it is shown that only the zero operator is a ring derivation on a semi-simple commutative Banach algebra with the carrier space without an isolated point, which is a generalization of a theorem of Nandakumar [4].

1. Lemmata.

LEMMA 1. *Let A be a commutative Banach algebra with the carrier space M_A. Suppose that D is a ring derivation on A. Then $(D(\alpha f))^\wedge=\alpha(D(f))^\wedge$ for every f in A and for every rational number α in the complex number field $\mathbf{C}$, where $\hat{}$ denotes the Gel'fand representation.*

PROOF. If α is a rational real number, then $D(\alpha f)=\alpha D(f)$ by standard argument. So we only show that $(D(if))^\wedge=i(D(f))^\wedge$, where i is the imaginary unit. For every f in A,

$$2fD(f)=D(f^2)=-D((if)^2)=-2ifD(if),$$

so we have $(D(f))^\wedge(x)=-i(D(if))^\wedge(x)$ for every x in M_A with $\hat{f}(x)\neq 0$. When $\hat{f}(x)=0$, choose g in A with $\hat{g}(x)\neq 0$. In the same way we have $(D(g))^\wedge(x)=-i(D(ig))^\wedge(x)$ and $(D(f+g))^\wedge(x)=-i(D(i(f+g)))^\wedge(x)$ since $(f+g)^\wedge(x)=\hat{f}(x)+\hat{g}(x)\neq 0$, so

$$(D(f))^\wedge(x)+(D(g))^\wedge(x)=-i(D(if))^\wedge(x)-i(D(ig))^\wedge(x).$$

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