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The Witten Laplacian on Negatively Curved Simply Connected Manifolds

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1. Introduction.

The Hopf conjecture states that the Euler characteristic of a compact Riemannian $2n$-manifold $\bar{M}$ of negative sectional curvature satisfies $(-1)^n\chi(\bar{M})>0$ [6]. Applying the Chern-Gauss-Bonnet theorem gives the conjecture for $n=1, 2$, for spaces of constant curvature, and for spaces of sufficiently pinched curvature [5]. Singer's idea of instead using the L^2 index theorem to establish the Hopf conjecture has been successfully carried out for Kähler manifolds by Gromov [18] (cf. [11]). It is worth noting that the first examples of negatively curved manifolds not admitting metrics of constant negative curvature are rather recent [20], [19].

Singer's method depends on the vanishing of L^2 harmonic forms (except in the middle dimension) on the universal cover of a compact negatively curved manifold, as explained in §4. This raised the question of such vanishing for arbitrary simply connected negatively curved manifolds. Anderson's paper [1] shows that such vanishing results are not possible without a pinching condition; however, his examples admit no compact quotient, so Singer's approach is not ruled out. One of our main results (Corollary 4.4) is that for one-forms vanishing occurs except in the pinching region ruled out by Anderson's examples. In general, we obtain vanishing results and hence $(-1)^n\chi(\bar{M})\geq 0$ (Theorem 4.5) for manifolds of pinched negative curvature, where the pinching constant is more relaxed than in previous work, e.g. [5].

The vanishing theorems depend upon Witten's deformation $\square_\tau$ of the Laplacian on forms on M [21]. In contrast to Witten's work, in which the Morse inequalities are recovered by letting the deformation parameter τ go to infinity, the vanishing theorems arise through the study of small deformations. Moreover, instead of deforming the Laplacian by a Morse function as in [21], we use the distance function to a point. The

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