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Homologically Trivial Self-Maps

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0. Introduction.

In the previous paper [2], we investigated the image of the homology representation $H_X: [X, X] \to \mathrm{End}(H_*(X))$ for a complex X of the form $S^n \cup e^{n+2} \cup e^{n+4}$ where $[X, X]$ denotes the set of homotopy classes of self-maps of X. This problem was equivalent to characterizing a triple of integers (d_1, d_2, d_3) which can be obtained from a self-map as its degrees.

In this paper we investigate the case $(d_1, d_2, d_3)=(0, 0, 0)$, namely, homologically trivial self-maps of X. The homotopy groups $\pi_{n+4}(A)$ and $\pi_{n+3}(A)$, $A=S^n \cup e^{n+2}$, play an important role for our purpose. Then some differences exist between the case $n=2$ and the others, so we concentrate on the case $n \geqq 3$ in this paper. Our method is to construct short exact sequences containing $H_X^{-1}(0)$ in the category of sets with distinguished elements. If X is a suspended complex, this category turns out to be the category of groups, so the results become more clear. Here we state some results from our theorems. Let X be a complex as above.

(1) If $Sq^2(e^n) \neq 0$ $(n \geqq 6)$, or $Sq^2(e^n) \neq 0$ and $Sq^4(e^n) \neq 0$ $(n=4, 5)$, then homologically trivial self-maps of X are also homotopically trivial.

(2) If $Sq^2(e^n)=0$ and $Sq^2(e^{n+2}) \neq 0$ $(n \geqq 4)$, then the same conclusion as (1) holds with an exceptional case in which there exists only one homotopically non-trivial but homologically trivial self-map.

(3) For $n=3$, the set of homologically trivial self-maps contains countablly infinite many homotopically non-trivial ones.

(4) Let ξ be an n-dimensional real vector bundle over CP^2 and $T(\xi)$ be the Thom complex of ξ $(n \geqq 5)$. Suppose the second Stiefel-Whitney class $w_2(\xi)=0$. Then, for self-maps of $T(\xi)$, homological triviality is equal to homotopical triviality.

Throughout this paper we use the same notations as ones in [2] for generators of homotopy groups of spheres, and the following:

$$A=S^n \cup e^{n+2}, \quad X=A \cup e^{n+4}, \quad \text{and} \quad Y=X/S^n=S^{n+2} \cup e^{n+4},$$

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