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Quadratic Residue Graph and Shioda Elliptic Modular Surface $S(4)$

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Dedicated to the memory of Professor M. Ishida*

0. Introduction

For each prime number $p\equiv 1 \pmod 4$, one attaches a graph without direction to the prime field $\mathbf{F}_p=\mathbf{Z}/p\mathbf{Z}$ by means of the Legendre symbol (cf. §1.1). This graph leads naturally to a rank two reflexive sheaf, denoted $\mathfrak{E}_p$, on the $(p-1)$-dimensional complex projective space $\mathbf{P}_{p-1}(\mathbf{C})$ (cf. [SEK 1], [SEK 2]). If $p=5$, then it coincides with the Horrocks-Mumford bundle (cf. [H-M]). The sheaf is both arithmetic and combinatorial in nature and it is seen that invariants of the graph are useful to describe the structure of the sheaf $\mathfrak{E}_p$. As a typical example, which is our main result in [SEK 2], the fourth Chern class $c_4(\mathfrak{E}_p)$ $(\in\mathbf{Z})$ is given by

$$c_4(\mathfrak{E}_p)=-40\mathcal{N}\,, \tag{C1}$$

where (cf. §1.1)

$$\mathcal{N}=\#\{I\subset\mathbf{F}_p \mid \#I=4 \text{ and whose graph is isomorphic to the square}\}\,.$$

The set is related explicitly to a $K3$ surface, denoted V, which is defined to be the locus of the following quadratic relations in the five dimensional projective space $\mathbf{P}_5(\mathbf{F}_p)$ with homogeneous coordinates $z_{\alpha\beta}$ $(1\le\alpha\le\beta\le 4)$ (cf. [SEK 1])

$$z_{\alpha\beta}^2+z_{\beta\gamma}^2=z_{\alpha\gamma}^2 \qquad (1\le\alpha<\beta<\gamma\le 4)\,. \tag{C2}$$

Now, it is known that the Shioda elliptic modular surface $S(4)$ of level 4 is birationally equivalent to a certain Kummer surface (cf. [Sh 3]). In §1 we see that V is biregularly equivalent to the Kummer surface. By a structure theorem of $S(4)$ (cf. [Sh 3]), the zeta function of $S(4)$ (and so that of V) is expressed by means of the Gaussian sum. Using this we give the following explicit form for (C1)

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