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Birational Classification of Curves on Irrational Ruled Surfaces

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1. Introduction.

Let S be a non-singular ruled surface with positive irregularity q and D an irreducible curve on S, which are defined over the field of complex numbers.

The purpose of this paper is to study pairs (S, D) from the view point of birational geometry. Two pairs (S, D) and (S_1, D_1) are said to be birationally equivalent if there exists a birational map $\varphi : S \to S_1$ such that the proper image $\varphi[D]$ of D by φ coincides with D_1. Such φ is said to be a birational transformation between pairs.

The pair (S, D) is said to be non-singular whenever both S and D are non-singular. In this case, define $P_m[D]$ to be $\dim H^0(S, \mathcal{O}(m(D+K)))$ $(m > 0)$ and $\kappa[D]$ to be the $K + D$ dimension of S, which is denoted by $\kappa(D + K, S)$, where K indicates a canonical divisor on S. Both $P_m[D]$ and $\kappa[D]$ are invariant under birational transformations between pairs. If every exceptional curve E of the first kind on S satisfies the inequality $E \cdot D \geq 2$ $(E \neq D)$, then (S, D) is said to be *relatively minimal* (cf. [I1], [Sa1]). Moreover, (S, D) is said to be *minimal*, if every birational map from any non-singular pair (S_1, D_1) into (S, D) turns out to be a morphism. It is easily shown that every minimal pair is relatively minimal. Since S is an irrational ruled surface, the Albanese map $\alpha : S \to \mathbf{Alb}(S)$ gives rise to a surjective morphism $\alpha : S \to \alpha(S) = B$, which is a curve of genus q. Let F denote a general fiber of $\alpha : S \to B$. Then the intersection number $D \cdot F$ coincides with the mapping degree of $\alpha|_D : D \to B$, which is denoted by $\sigma(D)$.

Every irrational ruled surface is obtained from a $\mathbf{P}^1$-bundle over B by successive blowing ups. Suppose that X is a $\mathbf{P}^1$-bundle and C a curve on X. Then the group $\mathrm{Num}(X)$ of numerical equivalent classes of divisors on X is a free abelian group generated by an infinite section Γ_∞ and a fiber $F_u = \Phi^{-1}(u)$ of the $\mathbf{P}^1$-bundle X where Φ is the projection (cf. [Ha], p. 370, Proposition 2.3). Then $C \equiv \sigma\Gamma_\infty + eF_u$ for some integers σ and e where the symbol $\equiv$ means numerical equivalence between divisors. Note that $\sigma = C \cdot F_u = \sigma(C)$. Let $b = -\Gamma_\infty^2$, which is said to be the degree of X. Moreover, let the multiplicities of all the singular points of C be denoted by $m_1, m_2, \cdots, m_r$ $(m_1 \geq m_2 \geq \cdots \geq m_r)$ where infinitely near singular points are

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