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Barnes' Double Zeta Function, the Dedekind Sum and Ramanujan's Formula

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1. Introduction

Let $\zeta_2(s; w; \omega_1, \omega_2)$ be the Barnes double zeta function [2], [9].

In the present paper, we show that the residue computation of the contour integral representation of $\zeta_2(s; 1; 1, \omega)$ yields

(1) the reciprocity formula of Apostol's generalized Dedekind sum [1] for rational ω, and

(2) Ramanujan's formula for values of Riemann zeta function at positive odd arguments [6], [8] as the limit case of the formula obtained for irrational ω. This shows that, in a sense, the Dedekind sum and Ramanujan's formula live on the same ground provided by the Barnes double zeta function.

As for (1), more generally, we shall derive the reciprocity formula for the Apostol-Rademacher Dedekind sum using more general ζ_2.

In [10], the authors investigated three kinds of Dedekind sums of Apostol and Apostol-Rademacher type, by computing values of Barnes' double zeta functions at non-positive integers and derived their reciprocity laws. Their method is algebraic. Our method is analytic.

In (2) the formula can be viewed as a limit case $x + i\omega \to i\omega$ (ω: irrational). This seems a new view point for Ramanujan's formula.

So, as for limit cases of (1), (2), we may think of a proof of the reciprocity formula of Gaussian sum [12], of Riemann's Fragmente [11] in which Riemann considered the limit cases of formulas in Jacobi [7], and of Dedekind's Erläuterungen [5] to it. This point of "limiting" view will be important for further investigation of the Dedekind sum and Ramanujan's formula.

Our method is very powerful. Barnes' multiple Riemann zeta functions of various types relate with Dedekind sums of various types.

In a subsequent paper, we shall consider Barnes' triple Riemann zeta function. Then, in particular, we can derive the formula to be called "triple term formula" for Apostol's Dedekind sum, which is different from Rademacher's for ordinary Dedekind sum.

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