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On Graded Rings, II (Z^n-graded rings)

Shiro GOTO and Keiichi WATANABE

Nihon University and Tokyo Metropolitan University

Introduction

Let k be a field, S an affine semigroup, i.e., a finitely generated additive submonoid of N^n, and $k[S]$ the semigroup ring of S over k. Then S is called normal if the ring $k[S]$ is integrally closed. (This condition does not depend on the field k. See Proposition 1, [10].) In [10] Hochster proved that $k[S]$ is a Macaulay ring if S is a normal semigroup and deduced from this fact that, if G is a torus over k and if G acts on a finite-dimensional vector space V over k rationally, then the ring A^G of invariants under the induced action of G on the symmetric algebra A of V is a Macaulay ring. (His proof of the above fact on semigroup rings depends on a certain result concerning the shellability of real polytopes.) Further in [18] Stanley studied the Hilbert functions of the algebra $k[S]$ and gave a criterion of $k[S]$ to be a Gorenstein ring in case S is a normal semigroup. It seems to be interesting to ask when the ring $k[S]$ is Macaulay (resp. Gorenstein) in case S is not necessarily normal.

The main purpose of our paper is to give a purely ring-theoretic proof of the Hochster's result on normal semigroups and, applying our way of proof further to arbitrary affine semigroups S, to find a criterion of the ring $k[S]$ to be Macaulay (resp. Gorenstein) in terms of S. Note that this was achieved by the authors and Suzuki [5] in case S is a simplicial monoid.

For this purpose we will develope a certain theory of graded rings and modules. Let H be a finitely generated free abelian group. By definition, an H-graded ring is a commutative Noetherian ring R together with a family $\{R_h\}_{h\in H}$ of subgroups such that $R=\bigoplus_{h\in H}R_h$ and $R_hR_g\subset R_{h+g}$ for all $h, g\in H$. Similarly an H-graded R-module is an R-module M for which there is given a family $\{M_h\}_{h\in H}$ of subgroups so that $M=\bigoplus_{h\in H} M_h$ and $R_hM_g\subset M_{h+g}$ for all $h, g\in H$. A homomorphism

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