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Formulae for the Riemann Zeta Function at Half Integers

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Introduction

Let a denote a positive integer, $\zeta(s)$ the Riemann zeta function and B_k be the k-th Bernoulli number, respectively. Then it is well known that

$$(1) \qquad \zeta(2a)=\frac{(-1)^{a-1}B_{2a}(2\pi)^{2a}}{2(2a)!}\,.$$

But practically nothing is known about the numerical nature of $\zeta(2a+1)$ except for the irrationality of $\zeta(3)$ proved by R. Apéry (see [6]). There is the Ramanujan's formula, which shed light on this problem, proved by A. P. Guinand [4], E. Grosswald [2], [3], and others. Recently, Y. Matsuoka [5] formulated and proved the Ramanujan's formula for the values of $\zeta(s)$ at half integers. And he got interesting expressions for $\zeta(1/2)\zeta(2a-1/2)$ and $\zeta(-1/2)\zeta(2a+1/2)$, where a is greater than 1.

In the present paper, by a similar method used in [5], we shall give generalizations of Matsuoka's results.

§1. Notations and results.

From now on, we assume that a is an integer greater than 1 and b is a non-negative integer. As usual, $\boldsymbol{N}$ and $\boldsymbol{Q}$ denote the set of natural numbers and the field of rational numbers, respectively. For any positive integer n, we put

$$g_{a,b}(n)=\sum_{\substack{klm\mid n\\(k,l,m)\in N^3}} k^{-b-1/2}l^{2a-1}m^{2a-b-3/2}\,.$$

Further we put

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