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On a Family of Continued-Fraction Transformations and Their Ergodic Properties

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Introduction

The simple continued-fraction expansion of real numbers is an important concept in the theory of numbers. And the continued-fraction expansion defined by Hurwitz is also important because it is the expansion by the nearest integers. These two continued-fraction expansions give rise to many interesting problems not only in the theory of numbers but also in ergodic theory. More precisely, many people, starting with Gauss and Lévy, treated endomorphisms from an interval into itself induced from these continued-fraction expansions, and obtain many interesting results; see, for example, Lévy [1], Kuzmin [2], Hurwitz [3], Shiokawa [5] and Nakada-Ito-Tanaka [6].

In this paper we treat a one-parameter family of continued-fraction expansions, which we shall call α-continued-fraction expansions. We note that these α-continued-fraction expansions reduce to the simple continued-fraction expansions in case $\alpha=1$ and to the continued-fraction expansions of Hurwitz in case $\alpha=1/2$. We treat the following three problems:

(1) To investigate the rate of approximation by the nth approximants.

(2) To determine the form of the density function of the invariant measure for the endomorphism induced from an α-continued-fraction expansion, which we shall call an α-continued-fraction transformation.

(3) To investigate how the ergodic properties of α-continued-fraction transformations change when the parameter α changes.

As for (1), we give in §2 the following result:

(i) In case $1/2 \leqq \alpha \leqq (\sqrt{5}-1)/2$, we have

$$\left|x-\frac{p_n(x,\alpha)}{q_n(x,\alpha)}\right| \leqq \frac{2}{\sqrt{5}}|q_n(x,\alpha)|^{-2}, \qquad |q_n(x,\alpha)| \geqq \left(\frac{\sqrt{5}-1}{2}\right)^{-n}.$$

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