TOKYO J. MATH.
VOL. 5, NO. 2, 1982

Eta-Function on S^{2n-1}

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Let Y be a compact oriented riemannian manifold of dimension $2n-1$, $\Omega^q(Y)$ be the space of all differential q-forms on Y and put $\Omega^{ev}(Y)=\bigoplus_{p=0}^{n-1}\Omega^{2p}(Y)$. Let $A: \Omega^{ev}(Y)\rightarrow\Omega^{ev}(Y)$ be a first order differential operator defined by

$$(1) \qquad A\phi=i^n(-1)^{p+1}(*d-d*)\phi \qquad (\phi\in\Omega^{2p}(Y))$$

where $i=\sqrt{-1}$, d is the exterior differential and $*$ is the Hodge duality operator. Then A is formally self-adjoint, elliptic and the squre A^2 is the Laplace operator $\Delta=d\delta+\delta d$, where δ is the formal adjoint of d. Therefore A is diagonalizable with real eigenvalues and, of course, the eigenvalues of A can be either positive or negative—they are square roots of the eigenvalues of Δ.

Now let G be a compact group of orientation preserving isometries on Y and suppose that A commutes with the action of G, then the λ-eigenspace E_λ of A is a finite dimensional G-module. In this situation, Atiyah-Patodi-Singer [4] defined the so-called "eta-function"

$$(2) \qquad \eta_A(g, s)=\sum_{\lambda\neq 0}(\mathrm{sign}\,\lambda)\,\mathrm{Tr}\,(g|E_\lambda)\cdot|\lambda|^{-s}$$

for any $g\in G$, where the summation is taken over all distinct eigenvalues of A and $g|E_\lambda$ is the transformation induced by g on E_λ.

For example, when Y is the circle S^1 and g is rotation through an angle θ, we have already known that

$$\eta_A(g, s)=-2i\cdot\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^s},$$

(see [4, p. 413]), and when Y is the 3-sphere S^3 and g is represented by the matrix $\begin{pmatrix}D(\theta_1) & 0\\ 0 & D(\theta_2)\end{pmatrix}$, where $D(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ is rotation of $\boldsymbol{R}^2$ by an angle θ, K. Katase calculated directly this η-function by determining the basis for the eigenspace of A (see [12]). On the other hand, J. J.

Received June 23, 1981