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On the Values of a Certain Dirichlet Series at Rational Integers

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It has known that the Riemann zeta function $\zeta(s)$ satisfies the relations

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)=2\zeta(3)\ ,$$

$$2\sum_{\nu=2}^{\infty}\frac{1}{\nu^n}\left(1+\frac{1}{2}+\cdots+\frac{1}{\nu-1}\right)=n\zeta(n+1)-\{\zeta(2)\zeta(n-1)+\cdots+\zeta(n-1)\zeta(2)\}$$
$$(n=3,\ 4,\ 5,\ \cdots)$$

(see [1], [2]). In this paper we prove the following

THEOREM. *Let $f(s)$ be the function defined by the Dirichlet series as*

(1) $$f(s)=\sum_{n=2}^{\infty}n^{-s}\sum_{k<n}k^{-1}\qquad(\mathrm{Re}\ s=\sigma>1)\ .$$

Then $f(s)$ is regular in the whole s-plane except at simple poles $s=0$ and $s=1-2a$ $(a=1, 2, 3, \cdots)$ with residues

$$\mathop{\mathrm{Res}}_{s=0}(f(s))=-\frac{1}{2}\ ,$$

$$\mathop{\mathrm{Res}}_{s=1-2a}(f(s))=-\frac{B_{2a}}{2a}\qquad(a=1,\ 2,\ 3,\ \cdots)\ ,$$

where B_n are Bernoulli numbers defined by $x/(e^x-1)=\sum_{n=0}^{\infty}(B_n/n!)x^n$, and a double pole $s=1$ with residue

$$\mathop{\mathrm{Res}}_{s=1}(f(s))=\gamma\qquad(\textit{Euler's constant})\ .$$

Further we have

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