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Examples of Simply Connected Compact Complex 3-folds

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In this note, we shall construct a series of compact complex manifolds $\{M_n\}_{n=1,2,3,\dots}$ of dimension 3 which are non-algebraic and non-kähler with the numerical characters $\pi_1(M_n)=0$, $\pi_2(M_n)=\boldsymbol{Z}$, $b_3(M_n)=4n$, $\dim H^1(M_n, \mathcal{O})\geqq n$, and $\dim H^1(M_n, \Omega^1)\geqq n$, where Ω^p is the sheaf of germs of holomorphic p-forms. These examples show, in particular, that, it is impossible to estimate $h^{p,q}(M)=\dim H^q(M, \Omega^p)$ of a compact complex manifold M in terms of its $(p+q)$-th Betti number, contrary to the case of dimension 2 or the case of kähler manifolds. To construct these examples, we employ a method of connecting two manifolds together to obtain a new one (see §§ 3 and 4).

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§ 1. We shall construct, in this section, a complex manifold X of dimension 3 with a projection

$$p: X \longrightarrow C$$

such that

(i) *$X-p^{-1}(0)$ is biholomorphic to the product of a primary Hopf surface $S_\alpha=\boldsymbol{C}^2-\{0\}/\langle\alpha\rangle$ and $\boldsymbol{C}^*=\boldsymbol{C}-\{0\}$ with $\alpha=\exp 2\pi ia$;*

(ii) *$p^{-1}(0)$ is simply connected, and is a union of two primary Hopf surfaces biholomorphic to $S_{\beta j}=\boldsymbol{C}^2-\{0\}/\langle\beta_j\rangle$ $(j=0, 1)$ with $\beta_j=\exp 2\pi ib_j$ which intersect each other normally in an elliptic curve, where $a\in\boldsymbol{C}$ is a fixed constant satisfying* $\operatorname{Im} a>0$, $b_0=a^{-1}$, *and* $b_1=(1-a)^{-1}$.

Let $a\in\boldsymbol{C}$ be a fixed number such that $\operatorname{Im} a>0$. Then $\alpha=\exp 2\pi ia$ satisfies $0<|\alpha|<1$. The multiplication $\xi\mapsto\alpha\xi$ for $\xi\in\boldsymbol{C}^*=\{\xi\in\boldsymbol{C}: \xi\neq 0\}$ defines a holomorphic automorphism of $\boldsymbol{C}^*$ and the quotient space $C=\boldsymbol{C}^*/\langle\alpha\rangle$ is an elliptic curve. Denote by $[\xi]$ the point on C corresponding to $\xi\in\boldsymbol{C}^*$. Take three copies W_j, $j=1, 2, 3$, of $\boldsymbol{C}^2$, on which we fix standard systems of coordinates (x_j, y_j). Let $X_j=W_j\times C$, and let $(x_j, y_j$:

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