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The Criteria of Kummer and Mirimanoff Extended to Include 22 Consecutive Irregular Pairs

Wilfrid KELLER and Günter LÖH

University of Hamburg
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Introduction

Let p be an odd prime and r an integer with $1\leqq r\leqq(p-3)/2$. If p divides the numerator of the Bernoulli number $B_{p-(2r+1)}$, or, equivalently, if $B_{p-(2r+1)}\equiv 0 \pmod p$, then the pair $(p, p-(2r+1))$ is said to be an irregular pair. For a given prime p, irregular pairs corresponding to consecutive integers r are called consecutive irregular pairs. The existence of consecutive irregular pairs associated with a prime p is intimately connected with the possibility of finding a nontrivial solution to the Fermat equation $x^p+y^p+z^p=0$ for the case that $(xyz, p)=1$. Thus, Wada [9] proved in 1979:

PROPOSITION 1. *If* $x^p+y^p+z^p=0$ *and* $(xyz, p)=1$, *then* $B_{p-(2r+1)}\equiv 0$ $(\mathrm{mod}\, p)$ *for* $r=1, 2, \cdots, 9$.

This proposition generalizes earlier results of Kummer (1857) and Mirimanoff (1905). For a history of the problem, see [8]. The condition imposed for a solution x, y, z to exist is a very stringent one. Since looking at all irregular pairs for primes $p<125000$ [10] and $1\leqq r\leqq 9$, not even two consecutive pairs are found, nor a single pair appears for $r=3$ or $r=6$. The following result, which, for sufficiently large values of p, is stronger than Wada's, was proved by Krasner [2] in 1934:

PROPOSITION 2. *If* $x^p+y^p+z^p=0$ *and* $(xyz, p)=1$, *and if* $p>n_0=(45!)^{88}$, *then* $B_{p-(2r+1)}\equiv 0 \pmod p$ *for* $r=1, 2, \cdots, k(p)$, *where* $k(p)=[\sqrt[3]{\log p}]$.

Note that the assumption of $p>n_0$ implies $k(p)\geqq[\sqrt[3]{\log n_0}]=22$. From Propositions 1 and 2 a number of summation criteria may be derived using the following result, proved by Ribenboim [7] in 1978:

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