TOKYO J. MATH.
VOL. 7, NO. 2, 1984

Omega Theorems for Divisor Functions

Shigeru KANEMITSU

Kyushu University
(Communicated by T. Mitsui)

Introduction

In what follows, ε denotes any positive number and c, with or without suffix, denotes a positive absolute constant, not necessarily the same at each occurence, unless otherwise specified.

Let $B_k(x)$ denote the k-th Bernoulli polynomial, $[x]$ the integral part of x, $P_k(x):=B_k(x-[x])$ the k-th periodic Bernoulli polynomial, $\sigma_r(n):=\sum_{d|n} d^r$ the sum of r-th powers of divisors of n, and define the basic functions $G_{a,k}(x)$ by

$$G_{a,k}(x):=\sum_{n\leq x^{1/2}} n^a P_k\left(\frac{x}{n}\right)$$

for real a and $k\in \boldsymbol{N}$.

As is well known, the summatory function of the divisor function $d(n):=\sigma_0(n)$ admits the asymptotic formula

$$\sum_{n\leq x} d(n)=x\log x+(2\gamma-1)x+\Delta(x)\ ,$$

where $\gamma=0.5772\ldots$ is the Euler(-Mascheroni) constant and the estimate $\Delta(x)=O(x^{1/2})$ is due to Dirichlet. The problem of estimating the error term $\Delta(x)$ carries the name of the Dirichlet divisor problem, the best known estimate being

$$\Delta(x)=O(x^{35/108+\varepsilon})$$

due to Kolesnik [16], and there is a conjecture that

$$\Delta(x)=O(x^{1/4+\varepsilon})\ . \tag{0.1}$$

In view of the well-known asymptotic relation (see e.g. MacLeod [19])

Received July 8, 1983