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On the Length of Modules over Artinian Local Rings

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Introduction

Let R be an Artinian local ring with the maximal ideal $\mathfrak{m}$. Then it is classical and well known that R is Gorenstein if and only if $l(\mathfrak{a})+l(0:\mathfrak{a})=l(R)$ for each ideal $\mathfrak{a}$ of R, where $0:\mathfrak{a}$ denotes the annihilator of $\mathfrak{a}$ and $l(M)$ denotes the length for an Artinian R-module M (cf. [1]). However in general we can say nothing about which is greater $l(\mathfrak{a})+l(0:\mathfrak{a})$ or $l(R)$ and so in this note we shall tackle with the question, introducing a certain invariant $t(M)$ for R-modules M (see Definition 1.1). In [2] the author defined the value $t(\mathfrak{a})=l(R/0:\mathfrak{a})/l(\mathfrak{a})$ for a non-zero ideal $\mathfrak{a}$ of R (for convenience we set $t(0)=1$) and gave the upper bound, the lower bound and other several properties of this value. Furthermore, he defined $t(M)$ for an Artinian R-module M and mentioned that similar results can be obtained, passing to idealization, for $t(M)$. In the present note, we will treat this value $t(M)$ directly.

At first we will prove inequalities $1/r(M)\leqq t(M)\leqq r(M)$ for any non-zero Artinian R-module M, where $r(M)$ denotes the dimension of the socle of M as a vector space over the residue field of R, that is $r(M)=\dim_{R/\mathfrak{m}}(0:\mathfrak{m})_M=l(0:\mathfrak{m})_M$. We will also consider the value $T(M)=\mathrm{Sup}_N\, t(N)$, where N runs over all R-submodules of M. Then from the above inequalities we obviously have $1\leqq T(M)\leqq r(M)$. And it will be proved that $T(M)=r(M)$ if and only if $r(M)=1$ if and only if $t(N)=1$ for each R-submodule N of M, which gives the classical result above mentioned when $M=R$. Next we will give an example which shows the above inequalities are the best possible in some sense. In the rest of the note we will study about when $T(R)=1$ or for what $\mathfrak{a}$, $t(\mathfrak{a})\leqq 1$.

Throughout this note let R denote an Artinian local ring with the maximal ideal $\mathfrak{m}$. An R-module will always means a finitely generated R-module.

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