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Calabi Lifting and Surface Geometry in S^4

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Introduction

Let $S^4(1)$ be the 4-dimensional unit sphere. Fix an orientation of $S^4(1)$. We denote by T_x the space of orthogonal complex structures compatible with the orientation of $T_x(S^4(1))$ and by $T=\bigcup_{x\in S^4(1)} T_x$ the fiber bundle over $S^4(1)$. T is called the *twistor space* of $S^4(1)$ and it is well-known that T is the 3-dimensional complex projective space P_3 and the projection of T onto $S^4(1)$ is the Hopf fibration. Let M be an orientable Riemannian surface isometrically immersed in $S^4(1)$ and J^1 a complex structure compatible with a fixed orientation of M. Then the normal space $N_x(M)$ admits an orthogonal complex structure J_x^2 defined by J_x^1 and the orientation of $S^4(1)$ and hence $J_x^1+J_x^2$ is an orthogonal complex structure of $T_x(S^4(1))$ for $x\in M$, where J_x^1 is the orthogonal complex structure of $T_x(M)$ given by J^1. Therefore we obtain a map of M into P_3 defined by $x\in M\to J_x^1+J_x^2\in P_3$. We call this map the *Calabi lifting* defined by the fixed orientation of $S^4(1)$ ([19]). Choosing the reverse orientation of $S^4(1)$, we have another Calabi lifting. We call the former the *positive* Calabi lifting and the latter the *negative* Calabi lifting. If M is an orientable surface in the 4-dimensional Euclidean space R^4, we can define the Gauss map of M into $S^2\times S^2$. We note that the Gauss map is constructed as follows: By the same argument as above, each point of M gives an orthogonal complex structure of R^4 compatible with the fixed orientation of R^4. Since the space of orthogonal complex structures of R^4 is S^2, we obtain a map of M into S^2. Changing the orientation of R^4, we get another map of M into S^2. It is easy to see that these maps of M into S^2 give the Gauss map of M into $S^2\times S^2$. From this point of view, we may regard Calabi lifting as "Gauss map".

In this paper, we investigate some relations between an isometric immersion into $S^4(1)$ and its Calabi lifting. In the sections 1, 2 and 3, we review the results of Chern [7] and Barbosa [2] on minimal surfaces

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