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A Certain Generalized Dedekind Sum

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Introduction

Let $[x]$ denote the greatest integer less than or equal to the real number x and $((x))=0$ or $x-[x]-\frac{1}{2}$ according as x is or is not an integer, respectively. L. Carlitz [3] studied some sums of the type $\sum_{k=1}^{p-1}\left[\frac{kq}{p}\right]^n$ and $\sum_{k=1}^{p-1}k^i\left[\frac{kq}{p}\right]^j$, where $(p, q)=1$, and obtained some reciprocity relations. These are related to the classical Dedekind sums which occur in the transformation formula for the logarithm of the Dedekind eta function. Recently, B.C. Berndt and L.A. Goldberg [2] studied the sums of the type $\sum_{k=1}^{p-1}(-1)^{\left[\frac{kq}{p}\right]}$, $\sum_{k=1}^{p-1}(-1)^k\left(\left(\frac{kq}{p}\right)\right)\left(\left(\frac{k}{p}\right)\right)$, etc., which arose in the transformation formula for the logarithm of the theta functions.

On the other hand, in order to decide whether the trigonometric sum $\sum_{k=1}^{p-1}\left|\cot\frac{k\pi}{p}\cdot\cot\frac{kq\pi}{p}\right|$ is a complete invariant for the isometric class of 3-dimensional lens spaces or not, we have to study some properties of the sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2\zeta^{\left[\frac{kp}{q}\right]q}$, where p is a prime number, $1\leqq q<p$ and ζ is a p^{th} root of unity. This sum is the classical Dedekind sum when $\zeta=1$ and this is the reason of the title of this paper.

Throughout this paper, let

$$H_n(p, q)=\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^n\zeta^{hq}$$

and

$$K_n(p, q)=\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^{n-1}\zeta^{\left[\frac{kp}{q}\right]q},$$

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