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Generalized Hilbert Transforms in Tempered Distributions

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Introduction

A Hilbert transform H of a function f on real field $\boldsymbol{R}$ is defined as:

$$Hf(x)=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} H_{\varepsilon,N}f(x)=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{f(x-t)}{t}dt \qquad (x\in\boldsymbol{R}).$$

The Hilbert transform H plays an important role in Fourier analysis. The properties of Hilbert transforms in the following Proposition are fundamental.

Let $L^p(\boldsymbol{R})$ be the class of all measurable functions f on $\boldsymbol{R}$ for which

$$\|f\|_{L^p}=\left(\int_{-\infty}^{\infty}|f(x)|^p dt\right)^{1/p}<\infty .$$

PROPOSITION. *Let p be a real number such that $1<p<\infty$. Then*

(i) [*existence*] *for any $f\in L^p(\boldsymbol{R})$,*

$$Hf(x)=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} H_{\varepsilon,N}f(x)$$

exists in the topology of $L^p(\boldsymbol{R})$,

(ii) [*boundedness*] *there exists a constant $C>0$ (independent of ε, N and f) such that*

$$\|Hf\|_{L^p}\leqq C\|f\|_{L^p}\ \ (\|H_{\varepsilon,N}f\|_{L^p}\leqq C\|f\|_{L^p}) \quad \text{for } all \quad f\in L^p(\boldsymbol{R}),$$

(iii) [*inversion formula*]

$$H(H(f))=-f \quad \text{for } all \quad f\in L^p(\boldsymbol{R}),$$

(iv) [*signum rule*]

$$(Hf)^{\wedge}=-i\,\mathrm{sgn}(x)\hat{f} \quad \text{for } all \quad f\in L^2(\boldsymbol{R}),$$

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