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A Completely Integrable Hamiltonian System of C. Neumann-type on the Complex Projective Space

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Introduction

The C. Neumann problem is a Hamiltonian system which describes the motion of a point on the sphere $S^{n-1}=\{x\in \boldsymbol{R}^n \mid \|x\|=1\}$ under the influence of a quadratic potential $U(x)=(1/2)\sum a_j x_j^2$, $a_1, \cdots, a_n \in \boldsymbol{R}$. It is shown by many authors that the C. Neumann problem is completely integrable (see [5, §1]). In [5], Ratiu showed that the C. Neumann problem is a Hamiltonian system on an adjoint orbit in a semidirect product of Lie algebras, and that its complete integrability follows entirely from Lie algebraic considerations.

In the present note, we define a C. Neumann-type problem on the complex projective space $P^{n-1}=\{[z] \mid z\in \boldsymbol{C}^n, \|z\|=1\}$ (see section 4). It is a Hamiltonian system which describes the motion of a point on P^{n-1} under the influence of a potential $U([z])=(1/2)\sum a_j|z_j|^2$. We then show, following Ratiu [5], that this system is a Hamiltonian system on an adjoint orbit in a semidirect product of Lie algebras (Theorem 3.4 and Proposition 4.1). As a consequence, we can prove that the C. Neumann-type problem on the complex projective space is completely integrable (Theorem 4.3).

§1. Hamiltonian actions and (co-)adjoint orbits.

In this section, we recall a few facts about symplectic geometry (for references, see [1], [2], [3], [4]). Let M be a symplectic manifold with symplectic structure Ω_M. Recall that Ω_M is a non-degenerate closed two-form on M. Each real-valued smooth (i.e., C^∞) function f on M generates a Hamiltonian vector field X_f on M which satisfies $X_f \lrcorner \Omega_M = df$. Let $C^\infty(M)$ be the space of real-valued smooth functions on M. The Poisson

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