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# On the Value of Dedekind Sums and Eta-Invariants for 3-Dimensional Lens Spaces

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## Introduction

Let $[x]$ denote the greatest integer less than or equal to the real number $x$, and $p$ and $q$ be relatively prime positive integers. It is a difficult problem to determine the exact value of the Dedekind sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^n$ except the trivial case: $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]=\frac{1}{2}(p-1)(q-1)$. In general, it is impossible to evaluate the sum for $n\geqq 2$ as a polynomial on $p$ and $q$. However, in case of $n=2$, we can express the sum as a finite sum with fewer terms using the sequence of remainders in the Euclidean algorithm for calculating the $\gcd(p, q)$ (see T. M. Apostol [1], p. 73).

On the other hand, instead of evaluating the sum, the reciprocity formulas which can be used as an aid in calculating the Dedekind sums have been studied. The only results we have known are:

$$(1)\qquad 6p\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^2+6q\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2=(p-1)(q-1)(2p-1)(2q-1)$$

and

$$(2)\qquad 4p(p-1)\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^3+4q(q-1)\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^3=(p-1)^2(q-1)^2(2pq-p-q+1)\ .$$

(See, for example, L. Carlitz [4] (1.7).)

There are some ways to prove these formulas (see, for example, T. M. Apostol and T. H. Vu [2], L. Carlitz [4], H. Rademacher and A. Whiteman [11], (3.5) and D. Zagier [12]). In the preceding paper [9], we introduced a lemma which was derived, for example, from Rademacher and Whiteman's method and is available to reduce almost all types of the generalized Dedekind sums to the sums of fewer terms. So, in this paper, we shall apply the lemma to evaluate the sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$.

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