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# Domains Integral over Each Underring

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## §1. Introduction and notation.

Let $R$ be a (commutative integral) domain with quotient field $K$. One has often studied $R$ by analyzing properties of its overrings (that is, the rings contained between $R$ and $K$). This article contributes to "dual" analyses via underrings. (We shall say that $B$ is an *underring* of $R$ in case $B$ is a subring of $R$ also having quotient field $K$.) Some studies of domains via underrings have already been done in [3], [4]. For instance, [4, Theorem 2.2] characterized the $R$ such that each underring of $R$ is seminormal. (It turned out that the same class of rings is characterized if "underring" is replaced by "subring" and/or "seminormal" is replaced by "a Euclidean domain." The rings $R$ in question are the overrings of $\boldsymbol{Z}$ (up to isomorphism) and the fields algebraic over a finite field.) This work completed earlier studies such as [5], [1, Proposition 2.11], and [3].

The domains characterized in [4, Theorem 2.2] may also be viewed as the domains each of whose underrings (resp., subrings) is integrally closed. One would nevertheless expect, rather generally, that requiring prescribed behavior of the underrings of $R$ would be less restrictive than requiring analogous behavior of all the subrings of $R$. To test this intuition and seek results with different flavor, we move to the "other extreme" from [4, Theorem 2.2]: we now ask to characterize the $R$ such that $R$ is integral over each of its underrings. It should be noted that the answer to *this* question changes if "underrings" is replaced by "subrings." The answers are given in Proposition 2.2, our main results (Theorem 2.3 and Corollary 3.6), and Remark 2.4 (a)-(c). It will be seen that consideration of *underrings* leads to certain global fields of positive

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