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On the Fundamental Units and the Class Numbers of Real Quadratic Fields II

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Introduction

Let M be a positive square-free integer and $\boldsymbol{Q}(\sqrt{M})$ be a real quadratic field with discriminant D. Denote by $h(M)$ and ε_M the class number and the fundamental unit of $\boldsymbol{Q}(\sqrt{M})$ respectively. After the works of Ankeny-Chowla-Hasse [1] and Hasse [5], there appeared several results about the lower bound of $h(M)$ with some conditions when $\varepsilon_M=(t+u\sqrt{D})/2$ is small (see Lang [6], Takeuchi [7] and Yokoi [9], [10], [11]). They used the basic result that the Diophantine equation $x^2-Dy^2=\pm 4m$ has no solutions in $\boldsymbol{Z}$ for $m<(t-2)/u^2$ if $N\varepsilon_M=1$, and for $m<t/u^2$ if $N\varepsilon_M=-1$. As a special case, we have $h(M)>(\log(D-1)/\log 4)-1$ for $M=(4C)^2+1$ $(C>1)$ from it. In this note, we also consider the same problem using continued fractions. We will get $h(M)>(\log D/\log 4)-1$ for $M=(C^s+\mu(C^t-\lambda))^2+4\lambda C^t$ with $s>t\geqq 1$, λ, $\mu=\pm 1$ if C is even and is not a power of 2. For these types of M with $t=1$, Bernstein [3], [4] gave the continued fractional expansion (c.f.e.) of $\sqrt{M}$ and the explicit representation of ε_M. The special case of them was mentioned in Yamamoto [8]. We also give ε_M explicitly for the above types of M and the lower bound of ε_M for another types of M from the c.f.e. of $\omega_0=(M_0+\sqrt{M})/2$ $(M_0<\sqrt{M}<M_0+2$, $M_0\equiv 1 \pmod 2)$. The lower bounds of ε_M were also given in [8] for sufficiently large M with several conditions. Then we investigate $h(M)$ for the above types of M and give the lower bounds with some conditions as mentioned above as a special case.

§1. Preliminaries.

In this section, we describe some basic properties of quadratic irrationals and ideals in real quadratic fields, which we will need in later

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