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Minimal Affine Boundaries of Convex Sets

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Introduction.

Let M be a compact convex set in a real locally convex linear topological space V and denote by $A(M)$ the set of restrictions on M of all real affine and continuous functionals in V, i.e. $f\in A(M)$ iff $f(tx+(1-t)y)=tf(x)+(1-t)f(y)$ for any $t\in \boldsymbol{R}$. Remind that a subset N of M is called an *end subset* of M iff it consists of points z that satisfy the following condition: z can not be represented as $z=\lambda x+\mu y$ with $\lambda>0$, $\mu>0$, $\lambda+\mu=1$, unless x and y belong to N. *Extreme points* of M are the points that are end subsets of M. Let $E(M)$ stand for the closure of extreme points of M. This is the smallest closed subset of M within which any positive element of $A(M)$ attains its minimum. Indeed, let $f\in A(M)$, $f>0$, and let $\min_{x\in M} f(x)=a<b=\min_{x\in E(M)} f(x)$. Since f is affine, the set $M\cap\{f(x)\geqq b\}$ is a compact convex set that contains $E(M)$ and consequently it contains also the closed convex hull of $E(M)$, i.e., it contains the whole set M according to the Krein-Milman's theorem (e.g. [1]). Hence $f(x)\geqq b>a$ on M, that is a contradiction. So every positive element of $A(M)$ attains its minimum within $E(M)$. If a closed subset N of M possesses the same property, then its closed convex hull $[\langle N\rangle]$ will coincide with M. In fact, if $[\langle N\rangle]\neq M$ we can find a positive continuous affine functional $f\in A(M)$ for which $f(x)\geqq a>0$ on $[\langle N\rangle]$ but $f(x_0)<a$ for some point $x_0\in M$ in contradiction with our supposition on N. But the equality $[\langle N\rangle]=M$ implies that $N\supset E(M)$ since the latter is the smallest closed subset of M for which $[\langle N\rangle]=M$ (e.g. [2]). Here we introduce n-dimensional analogues to the closure $E(M)$ of extreme points of a compact convex set M.

§1. Affine n-boundaries.

Denote by $A^n(M)$ the set of all n-tuples $(f_1, \cdots, f_n)$ of elements of

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