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On the Second Order Efficiency of Bootstrap Estimators of Sampling Distributions

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§ 1. Introduction.

Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random variables with unknown distribution function (d.f.) F contained in a set Θ of d.f.'s on the real line $\boldsymbol{R}$. Let $g_n(\cdot, F)$ be a d.f. on $\boldsymbol{R}$ parametrized by $F \in \Theta$, which will be considered to be a sampling d.f. of an appropriately normalized statistic based on the sample $\boldsymbol{X}_n=(X_1, X_2, \cdots, X_n)$ under F. We consider in this paper the estimation problem of $g_n(\cdot, F)$ based on the sample $\boldsymbol{X}_n=(X_1, \cdots, X_n)$. In particular, we discuss some asymptotic properties of the bootstrap estimator $\hat{g}_{n,B}=g_n(\cdot, \hat{F}_n)$ of $g_n(\cdot, F)$ where $\hat{F}_n$ is the empirical (sample) d.f. based on $\boldsymbol{X}_n=(X_1, \cdots, X_n)$. Consistency of $\hat{g}_{n,B}$ has been proved by Efron [6] and by Bickel and Freedman [4]. In Bickel and Freedman [3] and in Singh [8] Edgeworth type expansions of $\hat{g}_{n,B}$ for some typical g_n (the sampling d.f. of normalized sample mean and sample quantile) has been discussed. Beran [2] has proved that $\hat{g}_{n,B}$ is locally asymptotically minimax for estimating g_n under some smoothness conditions with respect to F. In this paper we prove the second order asymptotic efficiency of appropriately corrected version of $\hat{g}_{n,B}$ under conditions about $g_n(\cdot, F)$ similar to Assumption 1 or Assumption 1′ of Beran [2]. The concept of second order asymptotic efficiency in our case is essentially due to Akahira and Takeuchi [1]. We note that, in general, locally asymptotically minimax property does not imply second order efficiency as the following example shows: Let each X_i obey the distribution with density

$$f(x, \theta)=2^{-1} \exp(-|x-\theta|) \qquad (\theta \in \boldsymbol{R},\ x \in \boldsymbol{R}) .$$

In this case $\text{med}_{1\le i\le n} X_i$ are locally asymptotically minimax, but not second order asymptotically efficient for estimating $\theta \in \Theta$ (cf. Akahira and Takeuchi [1], p. 96).

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