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On the Algebraicity of the Ratio of Special Theta Constants

Hironori SHIGA

Chiba University

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In this paper we treat the special value of the Riemann θ function with characteristic ${}^t[a, b]$ $(a, b \in \boldsymbol{Q})$:

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)=\sum_{n\in \boldsymbol{Z}} \exp\{\pi i(n+a)^2\tau+2\pi i(n+a)(z+b)\} ,$$

where $z, \tau \in \boldsymbol{C}$ and $\operatorname{Im}\tau>0$. We show the following proposition:

PROPOSITION RA. *Suppose τ is an imaginary quadratic number with $\operatorname{Im}\tau>0$, then*

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(0, m\tau)\Big/\theta\begin{bmatrix} a' \\ b' \end{bmatrix}(0, \tau)$$

is an algebraic number for any a, b, a', b' of $\boldsymbol{Q}$ and any positive integer m, provided the denominator does not vanish.

This result plays a role of key stone in the forthcoming work concerning the modular form relative to the Picard modular group (it acts on 2-dimensional hyperball B^2).

PROOF. We divide the assertion in two parts:

(i) $\theta\begin{bmatrix} a \\ b \end{bmatrix}(0, \tau)\Big/\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)$ is algebraic,

(ii) $\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, m\tau)\Big/\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)$ is algebraic.

At first we show (i). Let us recall the transformation formula:

$$\text{(R-1)} \qquad \theta\begin{bmatrix} \tilde{\varepsilon}' \\ \tilde{\varepsilon}'' \end{bmatrix}(\tilde{z}, \tilde{\tau})=K(M, \varepsilon)\sqrt{c\tau+d}\cdot\exp\left(\frac{\pi icz^2}{c\tau+d}\right)\theta\begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix}(z, \tau) ,$$

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