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An Observation on the First Case of Fermat's Last Theorem

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Let p be an odd prime number. We consider Fermat's equation

$$x^p+y^p+z^p=0 \tag{1}$$

under the condition

$$xyz\not\equiv 0 \qquad (\text{mod}\ p)\ . \tag{2}$$

We abbreviate as $\mathrm{FLT}_1(p)$ the statement that the equation (1) has no solutions in integers under the condition (2). It is well-known that if p does not divide the (relative) class number of the cyclotomic field $L=\boldsymbol{Q}(\zeta)$, where ζ is a primitive p-th root of unity, then $\mathrm{FLT}_1(p)$ is true.

In the present paper, we study what we can say about $\mathrm{FLT}_1(p)$, supposing the relative class number of an imaginary subfield of L is not divisible by p. We prove the following:

THEOREM. *Suppose that $FLT_1(p)$ is not true, and let x, y, z be non-zero integers satisfying* (1) *and* (2). *Put $t=x/y$ and let*

$$H=\left\{t,\ \frac{1}{t},\ -\frac{1}{1+t},\ -(1+t),\ -\frac{t}{1+t},\ -\left(1+\frac{1}{t}\right)\right\}\ .$$

Let M be an arbitrarily fixed imaginary proper subfield of the cyclotomic field L. Put

$$\Phi(T)=N_{L/M}(T+\zeta)-N_{L/M}(T+\zeta^{-1})\ ,$$

where $N_{L/M}$ denotes the relative norm map from L to M. If p does not divide the relative class number h_M^- of the field M, then any number in the set H satisfies the congruence

$$\Phi(T)\equiv 0 \qquad (\text{mod}\ p)\ . \tag{3}$$

As an example, we consider the case M is a quadratic field $\boldsymbol{Q}(\sqrt{-p})$

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