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On the Power Series Coefficients of the Riemann Zeta Function

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§1. Introduction and the main result.

The Laurent expansion of the Riemann zeta function $\zeta(s)$ about the pole can be written in the form, in [2],

$$\zeta(s)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(s-1)^n \tag{1}$$

with

$$\gamma_n=\lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{\log^n k}{k}-\frac{\log^{n+1}N}{n+1}\right).$$

Here $\log^0 k$ mean 1 for all k including $k=1$. γ_0 is the well known Euler constant, and, for $n\geqq 1$, γ_n, sometimes called generalized Euler constants, have been studied by many authors ([1], Entry 13; or [3], p. 51). In this paper we shall give an asymptotic expansion of γ_n for arbitrary large n, which yields some interesting results on γ_n. They can be found in [4].

We begin by defining some notations. Let N be a nonnegative integer, and let n be a positive integer. In order to write our theorem, we need two functions $a=a(n)$ and $b=b(n)$ which are given by the following lemma.

LEMMA 1. *If $n>c_1$, where c_1 is a sufficiently large constant, then the system of the equations*

$$-(n+1)\frac{y}{x^2+y^2}+\frac{1}{2}\pi-\operatorname{Im}\psi(x+iy)=0\ , \tag{2}$$

$$-(n+1)\frac{x}{x^2+y^2}-\log 2\pi+\operatorname{Re}\psi(x+iy)=0\ , \tag{3}$$

with unknown x and y, satisfying $0<y<x$ and $n^{1/2}<x<n$, has a unique

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