TOKYO J. MATH.
VOL. 13, No. 2, 1990

On the Existence and Smoothness of Invariant Manifolds of Semilinear Evolution Equations

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(Communicated by S. T. Kuroda)

§1. Introduction.

Let us consider semilinear evolution equations in a Hilbert space X

$$du/dt = Lu + Nu\ , \qquad t>0\ . \tag{E}$$

Here L is the generator of an analytic semigroup and N is a nonlinear operator defined near 0. We suppose that the spectrum $\sigma(L)$ of L is divided into two parts $\sigma_1(L)$ and $\sigma_2(L)$ in such a way that

$$(\alpha_2 \equiv)\ \sup_{\sigma\in\sigma_2(L)} \operatorname{Re}\sigma < \inf_{\sigma\in\sigma_1(L)} \operatorname{Re}\sigma\ (\equiv \alpha_1)\ .$$

If N is identically zero, the eigenspace X_i, $i=1, 2$, corresponding to $\sigma_i(L)$ is invariant in the following sense: If an initial value x is contained in X_i then the solution $u(t, x)$ of (E) with the initial value x is also contained in X_i for $t>0$.

In this paper we are interested in the persistency of the invariance and smoothness of the manifolds X_i under small perturbation N. Let $N(x)$ be a C^k-mapping, $1\leqq k<\infty$, with $N(0)=0$. We first ask if there exists an invariant manifold M_i "near X_i", provided that $\|D_xN\|$ is small enough. (D_xN denotes the Fréchet derivative of $N(x)$ with respect to x.) If it does, we next ask if invariant manifolds are C^k.

The following facts have been known. See, e.g., [1-11, 14-17, 19-22].

(i) If $\inf_{\sigma\in\sigma_1(L)} \operatorname{Re}\sigma \geqq 0$, then an invariant C^k-manifold M_1 "near X_1" exists. It is called a center-unstable manifold. In particular, if $\inf_{\sigma\in\sigma_1(L)} \operatorname{Re}\sigma>0$ (resp. $\operatorname{Re}\sigma=0$ for $\sigma\in\sigma_1(L)$), then the manifold is called an unstable (resp. a center) manifold.

(ii) If $\sup_{\sigma\in\sigma_2(L)} \operatorname{Re}\sigma<0$, then an invariant C^k-manifold "near X_2" exists.

Received February 8, 1990