TOKYO J. MATH.
VOL. 14, No. 1, 1991

Spectral Flow and Maslov Index Arising from Lagrangian Intersections

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(Communicated by Y. Shimizu)

Dedicated to Professor Nobuhiko Tatsuuma on his 60-th birthday

§1. Introduction.

Let P be a $2n$-dimensional symplectic manifold with the symplectic structure ω, and L, L' be two Lagrangian submanifolds of P. We assume, in this paper, that L and L' intersect transversally with a non-empty intersection, and a smooth map

$$f : I\times I\to P\,, \qquad I=[0, 1]$$

is given with the following properties:

$$\begin{aligned} &f(\tau, 0)\in L\,, \quad f(\tau, 1)\in L' \qquad \text{for any} \quad \tau\in I\,, \\ &f(0, t)\equiv x\,, \quad f(1, t)\equiv y \qquad \text{for any} \quad t\in I\,, \end{aligned} \tag{1-1}$$

where $x, y\in L\cap L'$. Under these assumptions, two homotopic invariants arise. One is the spectral flow associated to a family of certain operators, and the other is the Maslov index of a curve which relates with the boundary conditions imposed on the operators in that family.

A. Floer has shown that these two are equal to relative Morse index from x to y, which is defined as the Fredholm index of a certain elliptic operator (see [F]).

Now let us explain the problem more precisely. Let g be a Riemannian metric on P which is adapted to the symplectic structure ω, that is, if we write $\omega(X, Y)=g(X, JY)$, for any vector fields X, Y on P, then J is an almost complex structure of P. Hence $J^2=-\mathrm{Id}$ and ${}^tJ=-J$ (${}^tJ$ is the transpose of J with respect to the metric g). We fix such a metric henceforth, and denote by ∇ the Riemannian connection of the metric g. Let Ω be a path space of P consisting of smooth paths $z: I\to P$ such that $z(0)\in L$ and $z(1)\in L'$. Consider, at least locally, a symplectic action functional $a: \Omega\to \boldsymbol{R}$ defined by

Received June 1, 1990