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Square-Free Discriminants and Affect-Free Equations

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§1. Square-free discriminants.

Unramified A_n-extensions of quadratic number fields are discussed by Uchida [5], [6] and Yamamoto [10]. Their results are closely related to the fact that there are infinitely many algebraic number fields K of degree n $(n>1)$ with the following properties:

1. The Galois group of $\bar{K}/Q$ is the symmetric group S_n, where $\bar{K}$ denotes the Galois closure of K/Q.
2. The discriminant of K is square-free.

It is the purpose of the present paper to discuss square-free discriminants and affect-free (affektlos) equations. We begin by proving the following theorem. The Galois closure of K/Q means the minimal Galois extension of Q which contains K.

THEOREM 1. *Let K denote an algebraic number field of degree n $(n\geq 1)$ and let $\bar{K}$ denote the Galois closure of K/Q. Suppose that the discriminant d of K is square-free. Then we have*:

1. *The Galois group of $\bar{K}/Q$ is the symmetric group S_n.*
2. *The Galois group of $\bar{K}/Q(\sqrt{d})$ is the alternating group A_n.*
3. *Every prime ideal is unramified in $\bar{K}/Q(\sqrt{d})$.*

PROOF. We may assume that $n>1$. Let G denote the Galois group of $\bar{K}/Q$. Then G is a transitive permutation group on $\{1, 2, \cdots, n\}$. Suppose that K has a subfield F such that

$$Q\subset F\subset K\,, \quad F\neq Q\,, \quad F\neq K\,.$$

Let d_F denote the discriminant of F. Then d is divisible by d_F^m, where $m=[K:F]$ ([1], Satz 39). Since $m>1$, by Minkowski's theorem we see that d cannot be square-free. This implies that G is primitive ([9], Theorem 7.4). Let p denote a prime number which divides d; by hypothesis d is exactly divisible by p. Then (van der Waerden [7]) the prime ideal decomposition of p (in K) is of the form

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