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On the $\wp$-Zero Value Function and the $\wp$-Zero Division Value Functions

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Introduction.

Let $\mathscr{H}$ be the upper half-plane $\{\tau \in \boldsymbol{C} \mid \operatorname{Im} \tau > 0\}$ and $\tau \in \mathscr{H}$. Let $\wp(u, \tau)$ denote the Weierstrass $\wp$-function with fundamental periods $(\tau, 1)$, (in more usual notation, it should be written $\wp(u; \tau, 1)$ or $\wp\left(u, \begin{pmatrix} \tau \\ 1 \end{pmatrix}\right)$). As is well known, $\wp(u, \tau)$ is a holomorphic function of two complex variables u, τ in a suitable region $\subset \boldsymbol{C} \times \mathscr{H}$, and the theorem of implicit function shows that, given a suitable region $D \subset \mathscr{H}$, there exists a holomorphic function $u_D(\tau)$ of $\tau \in D$ such that $\wp(u_D(\tau), \tau) = 0$ on D. This $u_D(\tau)$ is not uniquely determined by D. We shall show in this paper that there exists a unique analytic function u in $\mathscr{H}$, called "*$\wp$-zero value function*", such that every $u_D(\tau)$ are its branch on D (Theorem 1). This function u is a "many-valued modular form" in a sense to be indicated below. We shall show also in this paper the existence of another function $\mathfrak{p}_N$ of the same kind for an integer N greater than 1, which will be called "*N^{th} $\wp$-zero division value function*" (Theorem 2), and which is expected to have interesting arithmetical applications.

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NOTATIONS AND TERMINOLOGIES. In this paper, the symbol ":=" means that the expression on the right is the definition of that on the left. We put

$$\Gamma := SL_2(\boldsymbol{Z}), \quad U := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, for $z \in \boldsymbol{C}$, $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we set

$$Sz := \frac{az+b}{cz+d}, \qquad S : z := cz + d.$$

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