TOKYO J. MATH.
VOL. 16, NO. 2, 1993

On The Strong Ergodic Theorems for Commutative Semigroups in Banach Spaces

Hirokazu OKA

Waseda University

(Communicated by S. Suzuki)

1. Introduction.

This paper is concerned with the strong ergodic theorems for commutative semigroups.

Let C be a nonempty closed convex subset of a real Banach space X. A mapping $T: C \to C$ is said to be *asymptotically nonexpansive* if for each $n \geq 1$,

$$\| T^n x - T^n y \| \leq (1 + \alpha_n) \| x - y \| \qquad \text{for all } x, y \in C, \tag{1.1}$$

where $\lim_{n \to \infty} \alpha_n = 0$. In particular if $\alpha_n = 0$ for all $n \geq 1$, T is said to be *nonexpansive*. We denote by $F(T)$ the set of fixed points of a mapping T from C into itself. Let $\mathscr{T} = \{T(t) : t \geq 0\}$ be a family of mappings from C into itself. $\mathscr{T}$ is called an *asymptotically nonexpansive semigroup on C* if $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$, and there exists a function $\alpha(\cdot) : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ with $\lim_{t \to \infty} \alpha(t) = 0$ such that

$$\| T(t)x - T(t)y \| \leq (1 + \alpha(t)) \| x - y \| \tag{1.2}$$

for all $x, y \in C$ and $t \geq 0$. In particular, if $\alpha(t) = 0$ for all $t \geq 0$, then $\mathscr{T}$ is called a *nonexpansive semigroup on C*.

Baillon [2] and Bruck [3] proved the strong ergodic theorem for nonexpansive mappings in Hilbert spaces: let T be a nonexpansive mapping from C into itself and let $x \in C$. If $F(T)$ is nonempty and $\lim_{n \to \infty} \| T^n x - T^{n+k} x \|$ exists uniformly in $k = 0, 1, 2, \cdots$, then $\{T^n x : n \geq 1\}$ is strongly almost convergent as $n \to \infty$ to a point of y in $F(T)$, i.e.,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} T^{i+k} x = y \qquad \text{uniformly in } k = 0, 1, 2, \cdots.$$

The corresponding result for nonexpansive semigroups is the following: let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C. If $\bigcap_{t \geq 0} F(T(t))$ is nonempty and

Received May 22, 1992