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On Certain Infinite Series

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§0. Introduction.

The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},$$

where $n^s=\exp(s\,\mathrm{Log}\,n)$ and $\mathrm{Log}\,z$ denotes the principal branch of $\log z$. The series is locally uniformly convergent for $\mathrm{Re}(s)>1$, so that $\zeta(s)$ represents a regular function of s there. It is known that $\zeta(s)$ possesses an analytic continuation into the whole s-plane which is regular except for a simple pole at $s=1$ with residue 1 and that $\zeta(s)$ has the Laurent expansion at $s=1$ of the form

(1) $$\zeta(s)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma(n)(s-1)^n,$$

where

$$\gamma(n)=\lim_{N\to\infty}\left\{\sum_{k=1}^{N}\frac{\mathrm{Log}^n k}{k}-\frac{\mathrm{Log}^{n+1}N}{n+1}\right\}$$

for all values of n. In particular,

(2) $$\gamma=\gamma(0)=\lim_{N\to\infty}\left\{\sum_{k=1}^{N}\frac{1}{k}-\mathrm{Log}\,N\right\}$$

is called the Euler constant. The above expansion has been discovered independently by Briggs and Chowla [1] and a lot of mathematicians. It is also known that

(3) $$\zeta(0)=-\frac{1}{2},\quad \zeta(-2m)=0 \quad\text{and}\quad \zeta(1-2m)=-\frac{B_{2m}}{2m}$$

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