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Rauzy's Conjecture on Billiards in the Cube

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§1. Introduction.

We consider the billiards in the cube I^3 with $I=[0, 1]$. Let a particle start at a point $Q\in\bigcup_{i=1}^{3}F_i$ with constant velocity along a vector $v=(\alpha_1, \alpha_2, \alpha_3)$ and reflect at each face specularly, where $F_i:=\{(x_1, x_2, x_3) \mid x_i=0,\ 0\le x_j<1\ (j\ne i)\}$ $(i=1, 2, 3)$. Throughout this paper, we assume that

i) $\alpha_1, \alpha_2, \alpha_3>0$ are linearly independent over the field of rationals and

ii) the (forward) path of the particle never touches the edges of the cube.

If we label the two faces perpendicular to the x_i-ax as i and write down the label of the faces which the particle hits in order of collision, we have an infinite sequence $w(v, Q)$ of 1, 2, and 3. The complexity of an infinite sequence $w\in\{1, 2, 3\}^N$ is the function $p(n; w)$ defined as the number of distinct blocks $\in\{1, 2, 3\}^n$ appearing in w. In particular, we put $p(n; v, Q)=p(n; w(v, Q))$. Then the authors proved in [1] the following theorem conjectured by G. Rauzy [2–3] in 1981.

THEOREM. *Let v and Q satisfy the conditions* i) *and* ii). *Then the complexity of the sequence $w(v, Q)$ is given by*

$$p(n; v, Q)=n^2+n+1 \qquad (n\ge 1)\,.$$

The proof in [1] is based on a dynamical system associated with billiards in the cube. In this paper, we give an alternative proof, which is more elementary and independent of the ergodic arguments.

§2. The sequence $\{p_n\}_{n\ge1}$ and $\{q_n\}_{n\ge1}$.

By symmetry with respect to the faces, the word $w(v, Q)$ remains unchanged, if we replace the cube by the torus $\boldsymbol{R}^3/\boldsymbol{Z}^3$ and imagine that the particle does not reflect at the faces but passes through them. If we attach $i\in\{1, 2, 3\}$ to the intersection points of

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